下临界维数 Wiener sausage 相交时间的中偏差

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摘要 本文研究在下临界维数情形下 Wiener sausage 的相交时间, 应用新近提出的高阶矩方法和经典的 Feynman-Kac 半群方法, 得到该情况下 Wiener sausage 相交时间的中偏差.

关键词 Wiener sausage 相交局部时 高阶矩方法 中偏差

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1 引言

设 $\beta(s)$ 表示 $\mathbb{R}^d$ 空间中从原点出发的 Brown 运动, $\beta_1(s), \beta_2(s), \dots, \beta_p(s)$ 为 p 个相互独立且与 $\beta(s)$ 同分布的 Brown 运动. 由 $\{\beta(s), s \geq 0\}$ 产生的观察至时间 t 且以 r 为半径的 Wiener sausage 的体积定义为

$$W_r(t) := \bigcup_{0 \leq s \leq t} B_r(\beta(s)), \quad t \geq 0,$$

其中 $B_r(x)$ 是以 $x \in \mathbb{R}^d$ 为中心, 半径为 r 的开球.

Wiener sausage 的体积 $|W_r(t)|$ 是一个相当丰富的研究课题. 20 世纪 60 年代以来, 众多学者研究了它的各种极限行为. 例如, Wiener sausage 的大数定律^[1]、中心极限定理^[2]、Wiener sausage 的强逼近^[3, 4]和大偏差性质^[5-7]等. 近年来, Wiener sausage 相交轨道的极限行为成为学界研究的一个焦点 (参见文献 [8-10]). 一般地, 有两种方法来度量 p 个独立 Wiener sausage 的相交. 一种是计算相交部分的 Lebesgue 测度, 常把它称作相交体积; 另一种是通过下面的量来计算相交时间,

$$\mathcal{I}_r^{(p)}(t) := \int_{\mathbb{R}^d} \left(\int_{[0,t]^p} ds_1 \cdots ds_p \prod_{j=1}^p I_{[0,r]}(|\beta_j(s_j) - x|) \right) dx. \quad (1.1)$$

本文主要研究 Wiener sausage 相交情况的中偏差问题. 文献 [11] 得到了在下临界维数, 即 $p(d-2) < d$ 且 $d \geq 2$ 时 Wiener sausage 相交体积的中偏差. 本文研究该情况下相交时间的中偏差, 结果显示两者的中偏差速率函数极为相似. 主要结果是下面的定理.

定理 1.1 设 $b(t)$ 是一个正函数且满足

$$b(t) \rightarrow +\infty, \quad b(t) = o(t), \quad t \rightarrow +\infty. \quad (1.2)$$

(1) 当 $d = 2$ 且 $p \geq 2$ 时, 对任意的 $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{b(t)} \log \mathbb{P}\{\mathcal{I}_r^{(p)}(t) \geq \lambda t b(t)^{p-1}\} = -\frac{p}{2} (\pi r^2)^{-\frac{p}{p-1}} \kappa(2, p)^{-\frac{2p}{p-1}} \lambda^{\frac{1}{p-1}};$$

(2) 当 $d = 3$ 且 $p = 2$ 时, 对任意的 $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{b(t)} \log \mathbb{P}\{\mathcal{I}_r^{(p)}(t) \geq \lambda \sqrt{t b^3(t)}\} = -\left(\frac{3}{4}\right)^{\frac{4}{3}} (\pi r^3)^{-\frac{4}{3}} \kappa(3, 2)^{-\frac{8}{3}} \lambda^{\frac{2}{3}},$$

其中 $\kappa(d, p)$ 是满足下面 Gagliardo-Nirenberg 不等式的最优常数,

$$\|f\|_{2p} \leq C \|\nabla f\|_2^{\frac{d(p-1)}{2p}} \|f\|_2^{1-\frac{d(p-1)}{2p}}, \quad f \in W^{1,2}(\mathbb{R}^d) \quad (1.3)$$

且

$$W^{1,2}(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d); \nabla f \in L^2(\mathbb{R}^d)\}.$$

定理 1.1 的证明主要运用了近年来提出的高阶矩方法^[10, 12-14]. 这里不加证明地给出一个重要的 Gärtner-Ellis 型定理^[6, 定理 4].

定理 1.2 令 $\{Z_\epsilon\}$ 是一族非负的随机变量, $p \geq 1$ 为一个整数. 假设对任意的 $\theta > 0$, 下面的极限

$$\Psi(\theta) \equiv \lim_{\epsilon \rightarrow 0^+} \epsilon \log \sum_{m=0}^{\infty} \frac{(\theta \epsilon^{-1})^m}{m!} (\mathbb{E} Z_\epsilon^m)^{1/p}$$

存在. 若 $\Psi(|\theta|)$ 是本质光滑的, 则对任意的 $\lambda > 0$,

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \log \mathbb{P}(Z_\epsilon \geq \lambda) = -I_\Psi(\lambda),$$

其中

$$I_\Psi(\lambda) = p \sup_{\theta > 0} \{\theta \lambda^{1/p} - \Psi(\theta)\}.$$

由定理 1.2 可以看出, 为了证明定理 1.1, 我们只需要证明下面的高阶矩逼近结果. 为了方便, 我们记 $\mathcal{I}_r^{(p)}(t)$ 为 $\mathcal{I}_r(t)$.

命题 1.1 设 $b(t)$ 是一个正函数且满足 (1.2).

(1) 当 $d = 2$ 且 $p \geq 2$, 对任意的整数 $m \geq 1$ 和 $\theta > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t}\right)^{\frac{m}{p}} (\mathbb{E} \mathcal{I}_r^m(t))^{\frac{1}{p}} = \frac{1}{p} \left(\frac{2(p-1)}{p}\right)^{p-1} (\pi r^2 \theta)^p \kappa(2, p)^{2p}; \quad (1.4)$$

(2) 当 $d = 3$ 且 $p = 2$, 对任意的整数 $m \geq 1$ 和 $\theta > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t}\right)^{\frac{m}{4}} \sqrt{\mathbb{E} \mathcal{I}_r^m(t)} = \frac{8}{3} (\pi r^3 \theta)^4 \kappa(3, 2)^8. \quad (1.5)$$

命题 1.1 的证明分为上界和下界. 上界的证明主要依赖于 Wiener sausage 相交时间的高阶矩估计和高阶矩收敛性质. 下界的证明通过构造适当的 Feynman-Kac 半群和算子谱理论得到. 为了使文章脉络清晰, 本文将在第 2 节给出命题 1.1 的证明, 证明中用到的一些引理放在第 3 节完成证明.

2 命题 1.1 的证明

本节先不加证明地给出下面几个重要的引理.

引理 2.1 设 $t_1, t_2, \dots, t_a > 0$ 且满足 $t_1 + t_2 + \dots + t_a = t$, 则对任意的 $m \geq 1$,

$$(\mathbb{E}(\mathcal{I}_r(t))^m)^{\frac{1}{p}} \leq \sum_{\substack{k_1+k_2+\dots+k_a=m \\ k_1, k_2, \dots, k_a \geq 0}} \frac{m!}{k_1!k_2!\dots k_a!} \prod_{i=1}^a (\mathbb{E}(\mathcal{I}_r(t_i))^{k_i})^{\frac{1}{p}}. \quad (2.1)$$

进一步, 对任意的 $\theta > 0$,

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!} (\mathbb{E}(\mathcal{I}_r(t))^m)^{\frac{1}{p}} \leq \prod_{i=1}^a \sum_{m=0}^{\infty} \frac{\theta^m}{m!} (\mathbb{E}(\mathcal{I}_r(t_i))^m)^{\frac{1}{p}}. \quad (2.2)$$

引理 2.2 (1) 当 $d = 2$ 且 $p \geq 2$ 时,

$$\frac{1}{t^m} \mathbb{E} \mathcal{I}_r^m(t) \rightarrow (\pi r^2)^{mp} \mathbb{E}(\alpha([0, 1]^p))^m, \quad m = 1, 2, \dots, \quad t \rightarrow \infty; \quad (2.3)$$

(2) 当 $d = 3$ 且 $p = 2$ 时,

$$\frac{1}{t^{m/2}} \mathbb{E} \mathcal{I}_r^m(t) \rightarrow \left(\frac{4}{3} \pi r^3\right)^{2m} \mathbb{E}(\alpha([0, 1]^p))^m, \quad m = 1, 2, \dots, \quad t \rightarrow \infty, \quad (2.4)$$

其中 $\alpha([0, 1]^p)$ 表示 p 个独立 Brown 运动的相交局部时, 形式上可以写为

$$\alpha([0, 1]^p) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^p \int_0^1 \delta_x(\beta_j(s)) ds \right) dx.$$

引理 2.3 存在一个只依赖于 d 和 p 的常数 C 使得

(1) 当 $d = 2$ 且 $p \geq 2$ 时,

$$\sup_{t \geq 3} \mathbb{E} \exp\{C t^{-1/(p-1)} (\mathcal{I}_r(t))^{1/(p-1)}\} < \infty; \quad (2.5)$$

(2) 当 $d = 3$ 且 $p = 2$ 时,

$$\sup_{t \geq 3} \mathbb{E} \exp\left\{\frac{C}{t^{1/3}} \mathcal{I}_r(t)^{2/3}\right\} < \infty. \quad (2.6)$$

引理 2.4 设 f 是 $\mathbb{R}^d$ 上的有界连续函数, 则

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{b(t)} \log \mathbb{E} \left(\exp \left\{ \frac{b(t)}{C_r(d)t} \int_{\mathbb{R}^d} f \left(\sqrt{\frac{b(t)}{t}} x \right) \left(\int_0^t I_{[0,r]}(|\beta(s) - x|) ds \right) dx \right\} \right) \\ \geq \sup_{g \in \mathcal{H}_d} \left\{ \int_{\mathbb{R}^d} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla g(x), \nabla g(x) \rangle dx \right\}, \end{aligned} \quad (2.7)$$

其中 $C_r(d)$ 表示 $\mathbb{R}^d$ 中半径为 r 的球的体积,

$$\mathcal{H}_d = \{g \in L^2(\mathbb{R}^d); \|g\|_2 = 1, \|\nabla g\|_2 < \infty\}.$$

上界的证明 由于证明类似, 我们只证 $d = 2$ 的情形. 取 $s > 0$, $l_t = \frac{t}{b(t)}$. 由于 $b(t) = o(t)$, 故当 $t \rightarrow \infty$, $l_t \rightarrow \infty$, 利用引理 2.1, 我们可以得到

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t}\right)^{\frac{m}{p}} (\mathbb{E} \mathcal{I}_r^m(t))^{\frac{1}{p}} \leq \left(\sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t}\right)^{\frac{m}{p}} (\mathbb{E} \mathcal{I}_r^m(s l_t))^{\frac{1}{p}} \right)^{\left[\frac{t}{s l_t}\right]+1}.$$

由 (2.3), (2.5) 和控制收敛定理,

$$\lim_{t \rightarrow \infty} \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t} \right)^{\frac{m}{p}} (\mathbb{E} \mathcal{I}_r^m(s t_t))^{\frac{1}{p}} = \sum_{m=0}^{\infty} \frac{(\pi \theta r^2)^m}{m!} s^{\frac{m}{p}} (\mathbb{E} \alpha^m([0, 1]^p))^{\frac{1}{p}}. \quad (2.8)$$

故

$$\limsup_{t \rightarrow \infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t} \right)^{\frac{m}{p}} (\mathbb{E} \mathcal{I}_r^m(t))^{\frac{1}{p}} \leq \frac{1}{s} \log \sum_{m=0}^{\infty} \frac{(\pi \theta r^2)^m}{m!} s^{\frac{m}{p}} (\mathbb{E} \alpha^m([0, 1]^p))^{\frac{1}{p}}.$$

由文献 [15, 定理 2.1] 知, 当 $p(d-2) < d$ 且 $d \geq 2$ 时,

$$\lim_{t \rightarrow \infty} t^{-1} \log \mathbb{P}\{\alpha([0, 1]^p) \geq t^{d(p-1)/2}\} = -\frac{p}{2} \kappa(d, p)^{-4p/d(p-1)}.$$

利用定理 1.2 的逆定理 (类似于 Varadhan 积分引理, 参见文献 [15, 定理 5]) 将上述结果应用于 $d = 2$ 的情况, 取 $\epsilon = t^{-1}$, $Z_\epsilon = t^{-(p-1)} \alpha([0, 1]^p)$ 且 $I(\lambda) = \frac{p}{2} \kappa(2, p)^{-\frac{2p}{p-1}} \lambda^{\frac{1}{p-1}}$, 我们得到了下界估计

$$\begin{aligned} & \lim_{s \rightarrow \infty} \frac{1}{s} \log \sum_{m=0}^{\infty} \frac{(\pi \theta r^2)^m}{m!} s^{\frac{m}{p}} (\mathbb{E} \alpha([0, 1]^p))^{\frac{1}{p}} \\ &= \sup_{\lambda > 0} \left\{ \pi \theta r^2 \lambda^{\frac{1}{p}} - \frac{1}{2} \kappa(2, p)^{-\frac{2p}{p-1}} \lambda^{\frac{1}{p-1}} \right\} \\ &= \frac{1}{p} \left(\frac{2(p-1)}{p} \right)^{p-1} (\pi \theta r^2)^p \kappa(2, p)^{2p}. \end{aligned}$$

下界的证明 我们只证 $d = 2$ 的情形. 为了简便, 记

$$\mathcal{F}(\beta, t, x) = \int_0^t I_{[0, r]}(|\beta(s) - x|) ds, \quad \mathcal{F}(\beta_j, t, x) = \int_0^t I_{[0, r]}(|\beta_j(s) - x|) ds.$$

设 f 为 $\mathbb{R}^2$ 上的非负有界一致连续函数且 $\|f\|_q = 1$, 对任意的整数 $m > 1$, 由变量替换和 Hölder 不等式,

$$\begin{aligned} & \mathbb{E} \left(\int_{\mathbb{R}^2} f \left(\sqrt{\frac{b(t)}{t}} x \right) \mathcal{F}(\beta, t, x) dx \right)^m \\ &= \left(\frac{t}{b(t)} \right)^m \int_{\mathbb{R}^{2m}} \prod_{k=1}^m f(x_k) \mathbb{E} \left(\prod_{k=1}^m \mathcal{F} \left(\beta, t, \sqrt{\frac{t}{b(t)}} x_k \right) \right) dx_1 \cdots dx_m \\ &\leq \|f\|_q^m \left(\frac{t}{b(t)} \right)^m \left(\int_{\mathbb{R}^{2m}} \left(\mathbb{E} \left(\prod_{k=1}^m \mathcal{F} \left(\beta, t, \sqrt{\frac{t}{b(t)}} x_k \right) \right) \right)^p dx_1 \cdots dx_m \right)^{1/p} \\ &= \left(\frac{t}{b(t)} \right)^{\frac{p-1}{p}m} \left(\int_{\mathbb{R}^{2m}} \mathbb{E} \left(\prod_{j=1}^p \prod_{k=1}^m \mathcal{F}(\beta_j, t, x_k) \right) dx_1 \cdots dx_m \right)^{1/p} \\ &= \left(\frac{t}{b(t)} \right)^{\frac{p-1}{p}m} (\mathbb{E} \mathcal{I}_r^m(t))^{1/p}. \end{aligned} \quad (2.9)$$

因此,

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t} \right)^{m/p} (\mathbb{E} \mathcal{I}_r^m(t))^{1/p}$$

$$\begin{aligned} &\geq \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t} \right)^m \mathbb{E} \left(\int_{\mathbb{R}^2} f \left(\sqrt{\frac{b(t)}{t}} x \right) \mathcal{F}(\beta, t, x) dx \right)^m \\ &= \mathbb{E} \left(\exp \left\{ \theta \frac{b(t)}{t} \int_{\mathbb{R}^2} f \left(\sqrt{\frac{b(t)}{t}} x \right) \mathcal{F}(\beta, t, x) dx \right\} \right). \end{aligned}$$

由引理 2.4,

$$\begin{aligned} &\liminf_{t \rightarrow \infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t} \right)^{m/p} (\mathbb{E} \mathcal{I}_r^m(t))^{1/p} \\ &\geq \sup_{g \in \mathcal{H}_2} \left\{ \pi r^2 \theta \int_{\mathbb{R}^2} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \nabla g(x) \rangle dx \right\}. \end{aligned}$$

利用 Hölder 不等式, 对满足条件的 f 取上确界, 上式变为

$$\begin{aligned} &\sup_{g \in \mathcal{H}_2} \left\{ \pi r^2 \theta \left(\int_{\mathbb{R}^2} |g(x)|^{2p} dx \right)^{1/p} - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \nabla g(x) \rangle dx \right\} \\ &= \frac{1}{p} \left(\frac{2(p-1)}{p} \right)^{p-1} (\pi r^2 \theta)^p \kappa(2, p)^{2p}, \end{aligned} \quad (2.10)$$

这里最后一步来自于文献 [15, 引理 A.2]. □

3 引理的证明

本节将完成前面几个引理的证明. 为了证明引理 2.1, 先引入如下记号:

$$\begin{aligned} l_0 &= 0, \quad l_k = \sum_{i=1}^k t_i, \quad k = 1, 2, \dots, a-1, \quad l_a = t, \\ \mathcal{F}_k^x(\beta) &= \int_0^{l_k} I_{[0,r]}(|\beta(s) - x|) ds, \quad \mathcal{G}_k^x(\beta) = \int_0^{t_k} I_{[0,r]}(|\beta(s) - x|) ds, \quad k = 1, 2, \dots, a. \end{aligned}$$

记 θ 为 Brown 运动的推移算子, 容易看出

$$\mathcal{F}_a^x(\beta) \leq \sum_{i=1}^a \mathcal{G}_i^x(\beta \circ \theta_{l_{i-1}}).$$

引理 2.1 的证明 (2.2) 是 (2.1) 的简单推论, 我们只证 (2.1). 由 $\mathcal{I}_r^m(t)$ 的定义,

$$\begin{aligned} (\mathbb{E} \mathcal{I}_r^m(t))^{1/p} &= \left(\mathbb{E} \left(\int_{R^d} \prod_{j=1}^p \mathcal{F}_a^x(\beta_j) dx \right)^m \right)^{1/p} \\ &\leq \left(\mathbb{E} \left(\int_{R^d} \prod_{j=1}^p \sum_{i=1}^a \mathcal{G}_i^x(\beta_j \circ \theta_{l_{i-1}}) dx \right)^m \right)^{1/p} \\ &= \left(\mathbb{E} \left(\int_{R^{dm}} \prod_{k=1}^m \prod_{j=1}^p \sum_{i=1}^a \mathcal{G}_i^{x_k}(\beta_j \circ \theta_{l_{i-1}}) dx_1 \cdots dx_m \right) \right)^{1/p} \\ &= \left(\int_{R^{dm}} \mathbb{E}^p \left(\prod_{k=1}^m \sum_{i=1}^a \mathcal{G}_i^{x_k}(\beta \circ \theta_{l_{i-1}}) \right) dx_1 \cdots dx_m \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
 &= \left(\int_{R^{dm}} \mathbb{E}^p \left(\sum_{i_1, i_2, \dots, i_m} \prod_{k=1}^m \mathcal{G}_{i_k}^{x_k}(\beta \circ \theta_{l_{i_k-1}}) \right) dx_1 \cdots dx_m \right)^{1/p} \\
 &\leq \sum_{i_1, i_2, \dots, i_m} \left(\int_{R^{dm}} \mathbb{E}^p \left(\prod_{k=1}^m \mathcal{G}_{i_k}^{x_k}(\beta \circ \theta_{l_{i_k-1}}) \right) dx_1 \cdots dx_m \right)^{1/p}.
 \end{aligned}$$

注意到, 对于一组给定的取值于 1 到 a 之间的整数 $i_1, i_2, \dots, i_m$, 记 $k_1, k_2, \dots, k_a$ 分别为其中 $1, 2, \dots, a$ 出现的次数, 则 $k_1 + \dots + k_a = m$. 根据排列组合的理论, 我们知道这个对应值重复计算的次数为 $\frac{m!}{k_1! \cdots k_a!}$. 为了证明 (2.1), 我们将 $x_1, x_2, \dots, x_m$ 重新记为 $x_{1,1} \cdots x_{1,k_1} \cdots x_{a,1} \cdots x_{a,k_a}$, 只需证明

$$\int_{R^{dm}} \mathbb{E}^p \left(\prod_{i=1}^a \prod_{j=1}^{k_i} \mathcal{G}_i^{x_{i,j}}(\beta \circ \theta_{l_{i-1}}) \right) dx_{1,1} \cdots dx_{1,k_1} \cdots dx_{a,1} \cdots dx_{a,k_a} \leq \prod_{i=1}^a \mathbb{E} \mathcal{I}_r^{k_i}(t_i). \quad (3.1)$$

记 $\Omega_t = \sigma(\beta_s, s \leq t)$, 利用 Brown 运动的 Markov 性, 我们知道

$$\begin{aligned}
 \mathbb{E}^p \left(\prod_{i=1}^a \prod_{j=1}^{k_i} \mathcal{G}_i^{x_{i,j}}(\beta \circ \theta_{l_{i-1}}) \right) &= \left(\mathbb{E} \left(\mathbb{E} \left(\prod_{i=1}^a \prod_{j=1}^{k_i} \mathcal{G}_i^{x_{i,j}}(\beta \circ \theta_{l_{i-1}}) \mid \Omega_{l_{a-1}} \right) \right) \right)^p \\
 &= \left(\mathbb{E} \left(\prod_{i=1}^{a-1} \prod_{j=1}^{k_i} \mathcal{G}_i^{x_{i,j}}(\beta \circ \theta_{l_{i-1}}) \cdot \mathbb{E}_{\beta(l_{a-1})} \prod_{j=1}^{k_a} \mathcal{G}_a^{x_{a,j}}(\beta) \right) \right)^p \\
 &= \mathbb{E} \left(\prod_{h=1}^p \prod_{i=1}^{a-1} \prod_{j=1}^{k_i} \mathcal{G}_i^{x_{i,j}}(\beta_h \circ \theta_{l_{i-1}}) \cdot \prod_{h=1}^p \left(\mathbb{E}_{\beta_h(l_{a-1})} \prod_{j=1}^{k_a} \mathcal{G}_a^{x_{a,j}}(\beta_h) \right) \right).
 \end{aligned}$$

先计算关于变量 $x_{a,1}, \dots, x_{a,k_a}$ 的积分, 利用 Hölder 不等式和变量替换 $x'_{a,j} = x_{a,j} - \beta_h(l_{a-1})$, 由 Lebesgue 测度平移不变性可得

$$\begin{aligned}
 &\int_{R^{dk_a}} \prod_{h=1}^p \left(\mathbb{E}_{\beta_h(l_{a-1})} \prod_{j=1}^{k_a} \mathcal{G}_a^{x_{a,j}}(\beta) \right) dx_{a,1} \cdots dx_{a,k_a} \\
 &\leq \prod_{h=1}^p \left(\int_{R^{dk_a}} \left(\mathbb{E}_{\beta_h(l_{a-1})} \prod_{j=1}^{k_a} \mathcal{G}_a^{x_{a,j}}(\beta_h) \right)^p dx_{a,1} \cdots dx_{a,k_a} \right)^{1/p} \\
 &= \prod_{h=1}^p \left(\int_{R^{dk_a}} \left(\mathbb{E} \prod_{j=1}^{k_a} \mathcal{G}_a^{x'_{a,j}}(\beta_h) \right)^p dx'_{a,1} \cdots dx'_{a,k_a} \right)^{1/p} \\
 &= \int_{R^{dk_a}} \left(\mathbb{E} \prod_{j=1}^{k_a} \mathcal{G}_a^{x'_{a,j}}(\beta) \right)^p dx'_{a,1} \cdots dx'_{a,k_a} \\
 &= \int_{R^{dk_a}} \mathbb{E} \left(\prod_{h=1}^p \prod_{j=1}^{k_a} \mathcal{G}_a^{x'_{a,j}}(\beta_h) \right) dx'_{a,1} \cdots dx'_{a,k_a} \\
 &= \mathbb{E} \left(\int_{R^d} \prod_{h=1}^p \mathcal{G}_a^x(\beta_h) dx \right)^{k_a} \\
 &= \mathbb{E} \mathcal{I}_r^{k_a}(t_a).
 \end{aligned}$$

重复上面的过程即可证明 (3.1). □

引理 2.2 的证明 引理 2.2 是文献 [16, 定理 3.1] 的直接推论, 证明略. □

为了证明引理 2.3, 本节先给出下面的两个引理.

引理 3.1 对任意的整数 $m \geq 1$, $\mathbb{E}\mathcal{I}_r^m(t) \leq (m!)^p(\mathbb{E}\mathcal{I}_r(t))^m$.

证明 注意到

$$\begin{aligned}\mathbb{E}(\mathcal{I}_r^{(p)}(t))^k &= \mathbb{E}\left(\int_{(\mathbb{R}^d)^k} \prod_{i=1}^k \prod_{j=1}^p \left(\int_0^t I_{[0,r]}(|\beta_j(s) - y_i|) ds\right) dy_1 \cdots dy_k\right) \\ &= \int_{(\mathbb{R}^d)^k} \mathbb{E}^p\left(\prod_{i=1}^k \left(\int_0^t I_{[0,r]}(|\beta(s) - y_i|) ds\right)\right) dy_1 \cdots dy_k,\end{aligned}\quad (3.2)$$

记 $\sum_k$ 表示 $\{1, 2, \dots, k\}$ 上所有全排列. 由 Brown 运动的 Markov 性,

$$\begin{aligned}\mathbb{E}(\mathcal{I}_r^{(p)}(t))^k &\leq \int_{(\mathbb{R}^d)^k} \left(\sum_{\sigma \in \sum_k} \mathbb{E}\left(\int_{0 \leq s_1 \leq \dots \leq s_k \leq t} \prod_{i=1}^k I_{[0,r]}(|\beta(s_i) - y_{\sigma(i)}|) ds_1 \cdots ds_k\right)\right)^p dy_1 \cdots dy_k \\ &\leq (k!)^p \int_{(\mathbb{R}^d)^k} \left(\mathbb{E}\left(\int_{0 \leq s_1 \leq \dots \leq s_k \leq t} \prod_{i=1}^k I_{[0,r]}(|\beta(s_i) - y_i|) ds_1 \cdots ds_k\right)\right)^p dy_1 \cdots dy_k \\ &\leq (k!)^p \int_{(\mathbb{R}^d)^k} \left\{\mathbb{E}\left(\int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta(s_i) - y_i|) ds_1 \cdots ds_{k-1}\right.\right. \\ &\quad \times \left.\left.\mathbb{E}_{\beta(s_{k-1})} \int_0^t I_{[0,r]}(|\beta(s_k) - y_k|) ds_k\right)\right\}^p dy_1 \cdots dy_k \\ &= (k!)^p \int_{(\mathbb{R}^d)^k} \mathbb{E}\left\{\prod_{j=1}^p \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta_j(s_i) - y_i|) ds_1 \cdots ds_{k-1}\right. \\ &\quad \times \left.\prod_{j=1}^p \mathbb{E}_{\beta_j(s_{k-1})} \int_0^t I_{[0,r]}(|\beta_j(s_k) - y_k|) ds_k\right\} dy_1 \cdots dy_k.\end{aligned}$$

先计算关于 y_k 的积分, 利用 Hölder 不等式,

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbb{E}\left\{\prod_{j=1}^p \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta_j(s_i) - y_i|) ds_1 \cdots ds_{k-1}\right. \\ &\quad \times \left.\prod_{j=1}^p \mathbb{E}_{\beta_j(s_{k-1})} \int_0^t I_{[0,r]}(|\beta_j(s_k) - y_k|) ds_k\right\} dy_k \\ &= \mathbb{E}\left\{\prod_{j=1}^p \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta_j(s_i) - y_i|) ds_1 \cdots ds_{k-1}\right. \\ &\quad \times \left.\int_{\mathbb{R}^d} \prod_{j=1}^p \mathbb{E}_{\beta_j(s_{k-1})} \int_0^t I_{[0,r]}(|\beta_j(s_k) - y_k|) ds_k dy_k\right\} \\ &\leq E\left\{\prod_{j=1}^p \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta_j(s_i) - y_i|) ds_1 \cdots ds_{k-1}\right. \\ &\quad \times \left.\prod_{j=1}^p \left(\int_{\mathbb{R}^d} \left(\mathbb{E}_{\beta_j(s_{k-1})} \int_0^t I_{[0,r]}(|\beta_j(s_k) - y_k|) ds_k\right)^p dy_k\right)^{1/p}\right\}.\end{aligned}$$

再利用变量替换 $y'_k = y_k - \beta_j(s_{k-1})$ 和 Lebesgue 测度的平移不变性, 上式等于

$$\begin{aligned} & E \left\{ \prod_{j=1}^p \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta_j(s_i) - y_i|) ds_1 \cdots ds_{k-1} \cdot \int_{\mathbb{R}^d} \left(\mathbb{E} \int_0^t I_{[0,r]}(|\beta(s_k) - y_k|) ds_k \right)^p dy_k \right\} \\ &= E \left\{ \prod_{j=1}^p \int_{0 \leq s_1 \leq \dots \leq s_{k-1} \leq t} \prod_{i=1}^{k-1} I_{[0,r]}(|\beta_j(s_i) - y_i|) ds_1 \cdots ds_{k-1} \right\} \cdot \mathbb{E} \mathcal{I}_r^{(p)}(t). \end{aligned}$$

重复这个过程即可证明引理 3.1. □

引理 3.2 存在一个仅依赖于 d 和 p 的常数 C 使得对任意的 $t \geq 3$,

(1) 当 $d = 2$ 且 $p \geq 2$,

$$\mathbb{E}(\mathcal{I}_r^m(t)) \leq (m!)^{p-1} C^m t^m, \quad m = 1, 2, \dots \quad (3.3)$$

(2) 当 $d = 3$ 且 $p = 2$,

$$\mathbb{E}(\mathcal{I}_r^m(t)) \leq (m!)^{\frac{3}{2}} C^m t^{\frac{m}{2}}, \quad m = 1, 2, \dots \quad (3.4)$$

证明 由于证明是类似的, 我们只证 $d = 2$ 的情况. 记 $l = \frac{t}{m} + 1$, 由引理 2.1, 2.2 和 3.1, 我们有

$$\begin{aligned} (\mathbb{E} \mathcal{I}_r^m(t))^{\frac{1}{p}} &\leq \sum_{\substack{k_1+k_2+\dots+k_m=m \\ k_1, k_2, \dots, k_m \geq 0}} \frac{m!}{k_1! k_2! \cdots k_m!} (\mathbb{E}(\mathcal{I}_r^{k_1}(l)))^{\frac{1}{p}} (\mathbb{E}(\mathcal{I}_r^{k_2}(l)))^{\frac{1}{p}} \cdots (\mathbb{E}(\mathcal{I}_r^{k_m}(l)))^{\frac{1}{p}} \\ &\leq \sum_{\substack{k_1+k_2+\dots+k_m=m \\ k_1, k_2, \dots, k_m \geq 0}} \frac{m!}{k_1! k_2! \cdots k_m!} k_1! k_2! \cdots k_m! (\mathbb{E}(\mathcal{I}_r(l)))^{\frac{k_1}{p}} (\mathbb{E}(\mathcal{I}_r(l)))^{\frac{k_2}{p}} \cdots (\mathbb{E}(\mathcal{I}_r(l)))^{\frac{k_m}{p}} \\ &= \binom{2m-1}{m} m! (\mathbb{E} \mathcal{I}_r(l))^{m/p} \leq \binom{2m}{m} m! C_1^m t^{m/p} \\ &\leq (m!)^{\frac{p-1}{p}} C_2^m t^{m/p}. \end{aligned}$$

引理 3.2 得证. □

引理 2.3 的证明 由引理 3.2 和 Taylor 展式很容易得到引理 2.3, 略过. □

为了证明引理 2.4, 我们引入 $L^2(\mathbb{R}^d)$ 上的线性算子 T :

$$(T\xi)(x) = \mathbb{E}_x \left(\exp \left\{ \int_{\mathbb{R}^d} f \left(\sqrt{\frac{b(t)}{t}} y \right) \int_0^t I_{[0,r]}(|\beta(s) - y|) ds dy \right\} \xi(\beta(t)) \right), \quad \xi \in L^2(\mathbb{R}^d).$$

引理 3.3 T 是 $L^2(\mathbb{R}^d)$ 上的自伴算子.

证明 仍记

$$l_t = \frac{t}{b(t)}, \quad \mathcal{F}(\beta, t, y) = \int_0^t I_{[0,r]}(|\beta(s) - y|) ds.$$

对任意的 $\xi, \eta \in L^2(\mathbb{R}^d)$, 利用变量替换有

$$\begin{aligned} \langle \eta, T\xi \rangle &= \int_{\mathbb{R}^d} \eta(x) \mathbb{E}_x \left(\exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}} y) \mathcal{F}(\beta, t, y) dy \right\} \xi(\beta(t)) \right) dx \\ &= \int_{\mathbb{R}^d} \eta(x) \mathbb{E} \left(\exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}}(x + y)) \mathcal{F}(\beta, t, y) dy \right\} \xi(x + \beta(t)) \right) dx \\ &= \mathbb{E} \left(\int_{\mathbb{R}^d} \eta(x) \exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}}(x + y)) \mathcal{F}(\beta, t, y) dy \right\} \xi(x + \beta(t)) dx \right) \\ &= \mathbb{E} \left(\int_{\mathbb{R}^d} \eta(x - \beta(t)) \exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}}(x + y - \beta(t))) \mathcal{F}(\beta, t, y) dy \right\} \xi(x) dx \right). \end{aligned}$$

记 $\{\beta'(s) = -\beta(t) + \beta(t-s), 0 \leq s \leq t\}$, 则 $\{\beta'(s), 0 \leq s \leq t\} \stackrel{d}{=} \{\beta(s), 0 \leq s \leq t\}$. 上式等于

$$\begin{aligned} & \mathbb{E} \left(\int_{\mathbb{R}^d} \eta(x + \beta'(t)) \exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}}(x+y)) \mathcal{F}(\beta', t, y) dy \right\} \xi(x) dx \right) \\ &= \mathbb{E} \left(\int_{\mathbb{R}^d} \eta(x + \beta(t)) \exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}}(x+y)) \mathcal{F}(\beta, t, y) dy \right\} \xi(x) dx \right) \\ &= \int_{\mathbb{R}^d} \xi(x) \mathbb{E}_x \left(\exp \left\{ \int_{\mathbb{R}^d} f(\sqrt{l_t^{-1}}y) \mathcal{F}(\beta, t, y) dy \right\} \eta(\beta(t)) \right) dx \\ &= \langle T\eta, \xi \rangle. \end{aligned}$$

引理 3.3 得证. □

引理 2.4 的证明 为了简便, 我们记

$$\begin{aligned} l_t &= \frac{t}{b(t)}, \quad \Delta_i = [(i-1)l_t, il_t], \\ \mathcal{F}(\beta, x, i) &= \int_{s \in \Delta_i} I_{[0, r]}(|\beta(s) - x|) ds, \quad i = 1, 2, \dots \end{aligned}$$

定义 $L^2(\mathbb{R}^2)$ 上的算子半群如下: 对任意的 $\xi \in L^2(\mathbb{R}^2)$,

$$(T_t \xi)(x) = \mathbb{E}_x \left(\exp \left\{ \frac{1}{C_r(d)l_t} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}y) \int_0^{l_t} I_{[0, r]}(|\beta(s) - y|) ds dy \right\} \xi(\beta(l_t)) \right).$$

由引理 3.3 可知, T_t 是自伴算子.

设 g 是 $\mathbb{R}^2$ 上的无穷可微函数满足当 $x \notin [-M, M]^2$, $g(x) = 0$ 且 $\int_{\mathbb{R}^2} |g(x)|^2 dx = 1$,

$$\xi_t(x) = \frac{g(\sqrt{l_t^{-1}}x)}{\sqrt{\int_{\mathbb{R}^2} g^2(\sqrt{l_t^{-1}}y) dy}} = l_t^{-1/2} g(\sqrt{l_t^{-1}}x), \quad x \in \mathbb{R}^2.$$

记 $p_t(x)$ 是 $\beta(t)$ 的密度转移函数, 由 Brown 运动的 Markov 性, 有

$$\begin{aligned} & \mathbb{E} \exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=2}^{[b(t)]} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \\ &= \int_{\mathbb{R}^2} p_{l_t}(y) \mathbb{E}_y \exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]-1} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} dy \\ &\geq \frac{1}{\sup_y |g(y)|^2} \left\{ \int_{\mathbb{R}^2} g^2(\sqrt{l_t^{-1}}y) dy \right\} \int_{\mathbb{R}^2} p_{l_t}(y) \xi_t(y) \\ &\quad \times \mathbb{E}_y \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]-1} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \xi_t(\beta([b(t)] - 1)l_t) \right) dy. \end{aligned}$$

注意到

$$\begin{aligned} & \mathbb{E}_y \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]-1} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \xi_t(\beta([b(t)] - 1)l_t) \right) \\ &= \mathbb{E}_y \left(\mathbb{E}_y \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]-1} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \xi_t(\beta([b(t)] - 1)l_t) \mid \Omega_{([b(t)]-2)l_t} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E}_y \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]-2} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \right. \\
 &\quad \times \mathbb{E}_{\beta((\lfloor b(t) \rfloor - 2)l_t)} \left(\exp \left\{ \frac{1}{C_r(d)l_t} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, 1) dx \right\} \xi_t(\beta(l_t)) \right) \Big) \\
 &= \mathbb{E}_y \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]-2} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} T_t \xi_t(\beta((\lfloor b(t) \rfloor - 2)l_t)) \right) \\
 &= T_t^{[b(t)]-1} \xi_t(y).
 \end{aligned}$$

故

$$\mathbb{E} \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=2}^{[b(t)]} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \right) \geq \delta \int_{\mathbb{R}^2} \xi_t(y) T_t^{[b(t)]-1} \xi_t(y) dy,$$

其中

$$\delta = \frac{\inf_{t>0} l_t \inf_{y \in [-M\sqrt{l_t}, M\sqrt{l_t}]^2} \frac{1}{2\pi l_t} \exp\{-\frac{|y|^2}{2l_t}\}}{\sup_y |g(y)|^2}.$$

为了估计 $\int_{\mathbb{R}^2} \xi_t(y) T_t^{[b(t)]-1} \xi_t(y) dy$, 我们先考虑 T_t 的算子谱表示

$$T_t = \int_0^\infty \lambda dP_\lambda^t,$$

其中 $\{P_\lambda^t, \lambda \geq 0\}$ 是 T_t 的谱测度. 由于 $\int |g|^2 dx = 1$, 我们知道 $\langle P_\lambda^t \xi_t, \xi_t \rangle$ 是 $\mathbb{R}^2$ 上的概率测度, 利用 Jensen 不等式, 有

$$\langle \xi_t(x), T_t^{[b(t)]-1} \xi_t(x) \rangle = \int_0^\infty \lambda^{[b(t)]-1} d\langle P_\lambda^t \xi_t, \xi_t \rangle \geq \left(\int_0^\infty \lambda d\langle P_\lambda^t \xi_t, \xi_t \rangle \right)^{[b(t)]-1} = \langle \xi_t, T_t \xi_t \rangle^{[b(t)]-1}.$$

故

$$\liminf_{t \rightarrow \infty} \frac{1}{b(t)} \log \mathbb{E} \left(\exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=2}^{[b(t)]} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \right) \geq \liminf_{t \rightarrow \infty} \log \langle \xi_t, T_t \xi_t \rangle.$$

注意到

$$\begin{aligned}
 &\langle \xi_t, T_t \xi_t \rangle \\
 &= \left\{ \int_{\mathbb{R}^2} g^2(\sqrt{l_t^{-1}}y) dy \right\}^{-1} \int_{\mathbb{R}^2} g(\sqrt{l_t^{-1}}x) \\
 &\quad \times \mathbb{E}_x \left(\exp \left\{ \frac{1}{C_r(d)l_t} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}y) \mathcal{F}(\beta, y, 1) dy \right\} g(\sqrt{l_t^{-1}}\beta(l_t)) \right) dx \\
 &= l_t^{-1} \int_{\mathbb{R}^2} g(\sqrt{l_t^{-1}}x) \times \mathbb{E} \left(\exp \left\{ \frac{1}{C_r(d)l_t} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}(x+y)) \mathcal{F}(\beta, y, 1) dy \right\} g(\sqrt{l_t^{-1}}(x+\beta(l_t))) \right) dx \\
 &= \int_{\mathbb{R}^2} g(x) \mathbb{E} \left(\exp \left\{ \frac{1}{C_r(d)l_t} \int_{\mathbb{R}^2} f(x + \sqrt{l_t^{-1}}y) \int_0^{l_t} I_{[0,r]}(|\beta(s) - y|) ds dy \right\} g(x + \sqrt{l_t^{-1}}\beta(l_t)) \right) dx,
 \end{aligned}$$

利用变量替换 $s' = sl_t$, $y' = \sqrt{l_t^{-1}}y$, Brown 运动的标度不变性质和控制收敛定理, 我们很容易得到

$$\lim_{t \rightarrow \infty} \log \langle \xi_t, T_t \xi_t \rangle = \log \int_{\mathbb{R}^2} g(x) \mathbb{E}_x \left(\exp \left\{ \int_0^1 f(\beta(s)) ds \right\} g(\beta(1)) \right) dx.$$

考虑下面的 $L^2(\mathbb{R}^2)$ 上的算子半群:

$$T_t^f g(x) = \mathbb{E}_x \left(\exp \left\{ \int_0^t f(\beta(s)) ds \right\} g(\beta(t)) \right).$$

由 Feynman-Kac 公式知, T_t^f 的无穷小算子 $\mathcal{A}^f$ 是自伴的且满足

$$\mathcal{A}^f g = \frac{1}{2} \Delta g + fg, \quad g \in C^\infty(\mathbb{R}^2).$$

设 $\{P_\lambda^f, \lambda \geq 0\}$ 是 $\mathcal{A}^f$ 的谱测度. 由 $\int |g|^2 dx = 1$, 可以知道 $\langle P_\lambda^f g, g \rangle$ 是 $\mathbb{R}^2$ 上的概率测度, 利用 Jensen 不等式, 有

$$\begin{aligned} & \int_{\mathbb{R}^2} g(x) \mathbb{E}_x \left(\exp \left\{ \int_0^1 f(\beta(s)) ds \right\} g(\beta(1)) \right) dx \\ &= \langle g, T_1^f g \rangle = \int_0^\infty e^\lambda d\langle P_\lambda^f g, g \rangle \\ &\geq \exp \left\{ \int_0^\infty \lambda d\langle P_\lambda^f g, g \rangle \right\} = \exp \{ \langle g, \mathcal{A}^f g \rangle \} \\ &= \exp \left\{ \int_{\mathbb{R}^2} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \nabla g(x) \rangle dx \right\}. \end{aligned}$$

故

$$\begin{aligned} & \liminf_{t \rightarrow \infty} \frac{1}{b(t)} \log \mathbb{E} \left\{ \exp \left\{ \frac{1}{C_r(d)l_t} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \left(\int_0^t I_{[0,r]}(|\beta(s) - x|) ds \right) dx \right\} \right\} \\ &\geq \liminf_{t \rightarrow \infty} \frac{1}{b(t)} \log \mathbb{E} \left\{ \exp \left\{ \frac{1}{C_r(d)l_t} \sum_{i=1}^{[b(t)]} \int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x) \mathcal{F}(\beta, x, i) dx \right\} \right\} \\ &\geq \sup_{g \in \mathcal{H}_2} \left\{ \int_{\mathbb{R}^2} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \nabla g(x) \rangle dx \right\}. \end{aligned}$$

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Moderate deviations for the mutual intersection time of Wiener sausages in subcritical dimensions

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Abstract In this paper, we study the mutual intersection time of Wiener sausages in subcritical dimensions. Using the new high moment method and the Feynman-Kac semigroup, we obtain the corresponding moderate deviations.

Keywords Wiener sausage, intersection local times, high moment method, moderate deviations

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