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# The normal numbers of the fuzzy systems and their classes

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**Abstract** $\mathbb{R}$-fuzzy set is defined in this paper, which is regarded as the generalization of the Zadeh fuzzy set. By means of CRI method, some fuzzy systems are constructed by suitably using several kinds of $\mathbb{R}$-fuzzy sets as fuzzy inference antecedents, such as interpolation fuzzy system, Bernstein fuzzy system, Lagrange fuzzy system and Hermite fuzzy system. A notion of the normal number of the fuzzy system is defined here, we have shown that all fuzzy systems are able to be classified as three classes such as the normal fuzzy systems, the regular fuzzy systems and the singular fuzzy systems under the significance of the normal numbers of fuzzy systems. Finally, the generalized Bernstein polynomial is obtained by constructing Bernstein fuzzy system, it is proved that the generalized Bernstein polynomial is uniformly convergent in $C[a,b]$ under a weaker condition, and it is pointed out that there exist generalized Bernstein polynomials to be not convergent in $C[a,b]$ by use of constructing a counterexample.

**Keywords** $\bar{\mathbb{R}}$-fuzzy set, fuzzy system, the normal number of fuzzy system, universal approximation property for fuzzy system

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## 1 Introduction

It is not difficult to understand that fuzzy systems are a sort of representations to the uncertain systems, especially to the ones with fuzziness. It is considering of how to model on the systems with fuzziness that Zadeh defined fuzzy sets and designed modeling method by means of fuzzy inference so that it is possible to make good models to such uncertain systems (for example, see [1–26]). In [2], the fuzzy system $\underline{s}(x)$ is proved to be a certain interpolation function; especially under some conditions it is a piecewise interpolation function, that is, $s(x) \approx \underline{s}(x) = \sum_{i=0}^{n} A_i(x)y_i$, where $A_i$ $(i = 0, 1, \ldots, n)$ are a group of antecedents of the fuzzy inference rules as follows:

$$\text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i (i = 0, 1, \ldots, n), \tag{1}$$

and $y_i$ are the peak-points of the fuzzy sets $B_i (i = 0, 1, \ldots, n)$ as the consequents of the fuzzy inference rules (the peak-points mean $B_i(y_i) = 1, i = 0, 1, \ldots, n$). The input variable $x$ takes values in an input universe $X$ and the output variable $y$ takes values in an output universe $Y$.

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The statement mentioned above means a fuzzy system usually is an interpolation function. Whereas, from another point of view, we may find such meaning: given a piecewise interpolation function suiting with some conditions as $F_{n+1}(x)=\sum_{i=0}^{n}\psi_i(x)y_i : X\to Y$, there exists a group of fuzzy inference rules as (1) such that, based on the group of fuzzy inference rules, we are able to form a fuzzy system $\underline{s}$ by means of the CRI method, in which the connection $\underline{s}(x)$ between the input and the output of the system equates approximately or even accurately to $F_{n+1}(x)$ and the base functions $\psi_i(x)$ $(i=0,1,\ldots,n)$ in the interpolation meet that $\psi_i(x)=A_i(x), i=0,1,\ldots,n$.

This suggests us concern with such a problem that, in numerical approximation theory, there exist many interpolation methods (for example, see [27, 28]) that all are able to form an interpolation function, denoted by $F_{n+1}(x)$, and for such every method, whether can we make a group of fuzzy inference rules as the expression (1) such that, based on the group of fuzzy inference rules, a fuzzy system $\underline{s}$ may be constructed in which the connection $\underline{s}(x)$ between the input and the output of the system holds that $\underline{s}(x)\approx F_{n+1}(x)$ or $\underline{s}(x)=F_{n+1}(x)$? This is one of motivations of this paper.

It is well known that the fuzzy system is a kind of approximation to certain or uncertain system. So, it is very interesting to analysis holistically and describe quantitatively fuzzy systems from the view point of functional analysis, which is another motivation of this paper.

In order to characterize the fuzzy systems, a new kind of fuzzy set called $\mathbb{R}$-fuzzy set is introduced. Also the definition of normal number of fuzzy system is given. By inference method based on $\mathbb{R}$-fuzzy sets and the concept of normal number of fuzzy systems, some typical fuzzy systems are studied.

This paper is organized as follows: In section 2, the definition of $\mathbb{R}$-fuzzy sets is given. Also, the inference methods based on $\mathbb{R}$-fuzzy sets are studied. In section 3, the definition of normal number of the fuzzy system is given. It is pointed out that under the significance of the normal numbers of fuzzy systems, all fuzzy systems are able to be classified as three classes such as the normal fuzzy systems, the regular fuzzy systems and the singular fuzzy systems. In section 4, approximation of generalized Bernstein polynomials is discussed. Section 5 is conclusion.

## 2 $\mathbb{R}$-fuzzy sets

A Zadeh fuzzy set $A$ on a set $X$ is a mapping $\mu_A : X\to[0,1]$, where $\mu_A$ is called the membership function of $A$ and also denoted by $A(x)\triangleq\mu_A(x)$, for us being more convenient. We know very well that $[0,1]$ for the operations $\vee\triangleq\sup,\wedge\triangleq\inf$ and $x^c=1-x$ should form an F-lattice (see [29]). It is taking notice of that the generalized real number set, $\bar{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$, where $\mathbb{R}$ is the field of real numbers, for the operations $\vee\triangleq\sup,\wedge\triangleq\inf$ and $x^c=-x$ forms an F-lattice, too. $([0,1],\vee,\wedge,c)$ is obviously isomorphic with $(\bar{\mathbb{R}},\vee,\wedge,c)$.

Generally speaking, let $(L,\vee,\wedge)$ be a complete lattice. Every mapping $A : X\to L$ is called an $L$-fuzzy set (see [16, 30]), and the set of all such $L$-fuzzy sets is denoted by $L^X$. When $L=\bar{\mathbb{R}}$, the elements in the set $\bar{\mathbb{R}}^X=\{A|A : X\to\bar{\mathbb{R}}\}$ are naturally called $\bar{\mathbb{R}}$-fuzzy sets. In [31], the category of such generalized fuzzy sets with the degree set $L\triangleq\mathbb{R}\cup\{+\infty\}$ was researched, and the category is with good properties. For that reason mentioned above, we can and should generalize Zadeh fuzzy sets as follows.

**Definition 1.** Given a nonempty universe $X$, a generalized real-valued function $A : X\to\bar{\mathbb{R}}$ is called an $\bar{\mathbb{R}}$-fuzzy set. The set of all the $\bar{\mathbb{R}}$-fuzzy sets on $X$ is denoted by $\bar{\mathbb{R}}^X$. Particularly, when $A$ is a bounded function, $A$ is named a bounded fuzzy set, and the set of all the bounded fuzzy sets on $X$ is denoted by $BF(X)$.

**Definition 2.** A generalized binary real-valued function $\theta : \bar{\mathbb{R}}\times\bar{\mathbb{R}}\to\bar{\mathbb{R}}$ is called an $\bar{\mathbb{R}}$-fuzzy implication operations, if $\theta|_{[0,1]^2}$ is a fuzzy implication operator.

Because the generalized real-valued functions are able to take their values in $\{\pm\infty\}$, in order to avoid making mistakes, we emphasize some common stipulations as follows:

$$-\infty<+\infty;\quad x\in\mathbb{R}\Rightarrow-\infty<x<+\infty;\quad \pm\infty+(\pm\infty)=\pm\infty-(\mp\infty)=\pm\infty;$$
$$(\forall x\in\mathbb{R})(\pm\infty+x=\pm\infty=x-(\mp\infty));\quad (\forall x\in(0,+\infty))(\pm\infty\cdot x=\pm\infty);$$

$$(\forall x \in [-\infty, 0))(\pm\infty \cdot x = \mp\infty); \quad (\forall x \in \mathbb{R})(\pm\infty \cdot 0 = 0 = x/\pm\infty);$$
$$|\pm\infty| = +\infty; \quad (\pm\infty)\cdot(\pm\infty) = +\infty; \quad (\pm\infty)\cdot(\mp\infty) = -\infty.$$

Sometimes, $+\infty$ is simply denoted as $\infty$. Be carefully, $\pm\infty - (\pm\infty)$, $(\pm\infty) + (\mp\infty)$, and so on are insignificant.

**Example 1.** Let $\theta \triangleq \cdot$, i.e., $\theta : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}, (x, y) \mapsto \theta(x, y) = x \cdot y$. Then $\theta$ is an $\bar{\mathbb{R}}$-fuzzy implication operation. And let $\theta \triangleq \wedge : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}, (x, y) \mapsto \theta(x, y) = x \wedge y$. Then $\theta$ is also an $\bar{\mathbb{R}}$-fuzzy implication operation.

Now we again review the single input and single output fuzzy system. With the input universe $X = [a, b] \subset \mathbb{R}$ and the output universe $Y = [c, d] \subset \mathbb{R}$. Suppose we know a group of the input-output data $\{(x_i, y_i)|i = 0, 1, \ldots, n\}$ with the partitions on the universes $X$ and $Y$, as $a = x_0 < x_1 < \cdots < x_n = b$ and $c = y_{k_0} < y_{k_1} < \cdots < y_{k_n} = d$, respectively, where $k_i = \sigma(i)$ and the $\sigma$ is an $(n+1)$-ary permutation:

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & n \\ k_0 & k_1 & \cdots & k_n \end{pmatrix}.$$

Let $\Delta y_{k_i} = y_{k_{i+1}} - y_{k_i}, i = 0, 1, \ldots, n-1$, and $\Delta y_{k_n} = \frac{1}{n}\sum_{i=0}^{n-1} \Delta y_{k_i}$. Then we construct the bounded fuzzy sets $A_i \in BF(X)$ $(i = 0, 1, \ldots, n)$ as fuzzy inference antecedents and the $\bar{\mathbb{R}}$-fuzzy sets $B_i \in \bar{\mathbb{R}}^Y (i = 0, 1, \ldots, n)$ as fuzzy inference consequents, where the antecedent bounded fuzzy sets hold the normalizing condition: $\sum_{i=0}^{n} A_i(x) \equiv 1$. And for making the fuzzy inference consequent $\bar{\mathbb{R}}$-fuzzy sets $B_i \in \bar{\mathbb{R}}^Y (i = 0, 1, \ldots, n)$, we firstly form a group of the triangle wave fuzzy sets as $\bar{B}_{k_i} \in \mathcal{F}(Y)$ $(i = 0, 1, \ldots, n)$. Then the $\bar{B}_i$ $(i = 0, 1, \ldots, n)$ are expanded to be as the $\bar{\mathbb{R}}$-fuzzy sets $B_i \in \bar{\mathbb{R}}^Y (i = 0, 1, \ldots, n)$ as the following $B_i = \bar{B}_i/(\Delta y_i)^2, i = 0, 1, \ldots, n$, that is, $B_i(y) = \bar{B}_i(y)/(\Delta y_i)^2, y \in Y, i = 0, 1, \ldots, n$. It is not difficult to understand that every $1/(\Delta y_i)^2$ plays such a role that every $\bar{B}_i$ is weighted by $1/(\Delta y_i)^2$ to become a $B_i$, $i = 0, 1, \ldots, n$. Let $\{y_i\}_{\bar{\mathbb{R}}}$ $(i = 0, 1, \ldots, n)$ be an $\bar{\mathbb{R}}$-singleton, which means the truth value set of singletons is generalized from $\{0, 1\}$ to $\{-\infty, +\infty\}$, i.e.,

$$(\forall y \in Y)\left(\{y_i\}_{\bar{\mathbb{R}}}(y) = \begin{cases} +\infty, y = y_i \\ -\infty, y \neq y_i \end{cases}\right), \ i = 0, 1, \ldots, n.$$

According to the fuzzy inference rules (1), we choose the $\bar{\mathbb{R}}$-fuzzy implication operator as $\theta \triangleq \cdot$, then

$$R_i(x, y) = A_i(x) \cdot B_i(y), \quad i = 0, 1, \ldots, n. \tag{2}$$

In [32], an idea of weighted fuzzy inference was discussed. Here we will use such weighted fuzzy inferences to deal with our problem. As a matter of fact, these $\bar{\mathbb{R}}$-fuzzy relations $R_i$ obtained in (2) are aggregated by the weighted form to become a whole $\bar{\mathbb{R}}$-fuzzy relations on all rules as expression as $R(x, y) \triangleq \sum_{i=0}^{n} w_i R_i(x, y) = \sum_{i=0}^{n} w_i A_i(x) B_i(y), \quad i = 0, 1, \ldots, n$. These $w_i$ $(i = 0, 1, \ldots, n)$ represent for the portions of the $\bar{\mathbb{R}}$-fuzzy inference relations $R_i (i = 0, 1, \ldots, n)$ to occupy in the whole $\bar{\mathbb{R}}$-fuzzy inference relations $R$, respectively. Here we take that

$$w_i = \Delta y_i / \sum_{k=0}^{n} \Delta y_k, \ i = 0, 1, \ldots, n. \tag{3}$$

For the use of the $\bar{\mathbb{R}}$-fuzzy inference relation, we denote

$$M \triangleq \sup\{R(x, y)|(x, y) \in [a, b] \times [c, d]\}, \quad m \triangleq \inf\{R(x, y)|(x, y) \in [a, b] \times [c, d]\}.$$

For the requirements in what follows, we introduce a new concept, quasi-interpolation functions. So-called a quasi-interpolation function $F_{n+1}(x)$ means it holding the conditions as follows:

1) $F_{n+1}(x)$ is a linear combination of the group of basis functions $\{A_i(x), i = 0, 1, \ldots, n\}$, i.e., there exists a group of real numbers $\{c_i \in \mathbb{R}|i = 0, 1, \ldots, n\}$ such that $F_{n+1}(x) = \sum_{i=0}^{n} A_i(x) c_i$;

2) $c_i = y_i, i = 0, 1, \ldots, n$.

Clearly we know that, only with the above two conditions, well-known interpolation condition, $F_{n+1}(x_i) = y_i, i = 0, 1, \ldots, n$, cannot be met yet. It is easy to see that, if the group of basis functions $\{A_i(x)\}_{i=0}^n$ satisfy Kronecker condition:

$$A_i(x_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \end{cases} \qquad i, j = 0, 1, \ldots, n,$$

then that $F_{n+1}(x) = \sum_{i=0}^n A_i(x) y_i$ holds the interpolation condition.

**Theorem 1.** By still using notations and notions mentioned above, based on the group of fuzzy inference rules as (1), a fuzzy system $\underline{s}$ is obtained by means of CRI method is approximately equal to a quasi-interpolation function in which its basis functions are just the bounded fuzzy sets $A_i(x)$, as the following

$$\underline{s}(x) \approx F_{n+1}(x) \triangleq \sum_{i=0}^{n} A_i(x) y_i. \tag{4}$$

*Proof.* By CRI method, from the $\bar{\mathbb{R}}$-fuzzy inference relation $R$, an $\bar{\mathbb{R}}$-fuzzy transformation "$\circ$" is induced as follows:

$$\circ : \bar{\mathbb{R}}^X \longrightarrow \bar{\mathbb{R}}^Y, A \mapsto B \triangleq \circ(A) = A \circ R, B(y) = \bigvee_{x \in X} (A(x) \wedge R(x, y)), y \in Y. \tag{5}$$

For any given $x^* \in X$, denoting

$$A^*(x) \triangleq \begin{cases} M, & x = x^*; \\ m, & x \neq x^*, \end{cases}$$

and being substituted in (5), we get a fuzzy inference consequent $B^* \in \bar{\mathbb{R}}^Y$ as follows:

$$B^*(y) = (A^* \circ R)(y) = \bigvee_{x \in X} (A^*(x) \wedge R(x, y)) = \sum_{i=0}^{n} w_i A_i(x^*) B_i(y). \tag{6}$$

If $\int_Y |yB^*(y)| \mathrm{d}y < \infty$, $\int_Y |B^*(y)| \mathrm{d}y < \infty$ and $\int_Y B^*(y) \mathrm{d}y \neq 0$, we let $y^* \in Y$ with respect to $x^*$, i.e., $y^* = \int_Y yB^*(y)\mathrm{d}y / \int_Y B^*(y)\mathrm{d}y$, by using the barycenter method. Because $x^*$ is arbitrary, above expression should be generalized as follows:

$$y = \int_Y yB(y)\mathrm{d}y \Big/ \int_Y B(y)\mathrm{d}y, \tag{7}$$

where $B(y) = \sum_{i=0}^n w_i A_i(x) B_i(y)$. In accordance with the significance of Riemannian Sum in the definition of definite integral, we have

$$y = \frac{\int_Y B(y) y \mathrm{d}y}{\int_Y B(y)\mathrm{d}y} \approx \frac{\sum_{i=0}^n B(y_{k_i}) y_{k_i} \Delta y_{k_i}}{\sum_{i=0}^n B(y_{k_i}) \Delta y_{k_i}} = \frac{\sum_{i=0}^n y_i B(y_i) \Delta y_{k_i}}{\sum_{i=0}^n B(y_i) \Delta y_{k_i}}. \tag{8}$$

From $\sum_{i=0}^n A_i(x) \equiv 1$, we have that $\underline{s}(x) \approx F_{n+1}(x) \triangleq \sum_{i=0}^n A_i(x) y_i$.

# 3 Normal numbers of fuzzy systems

## 3.1 Normal numbers of fuzzy systems

In this section, we will discuss some analysis properties of fuzzy systems from functional analysis point of view. The group of $\bar{\mathbb{R}}$-fuzzy inference rules describing the uncertain system is as (1), that is, $A_i \to B_i$, $i = 0, 1, \ldots, n$, where $A_i \in \bar{\mathbb{R}}^X$ and $B_i \in \bar{\mathbb{R}}^Y$, $i = 0, 1, \ldots, n$. We let that

$$\mathcal{A} \triangleq \{A_0, \ldots, A_n\} \subset \bar{\mathbb{R}}^X; \mathcal{B} \triangleq \{B_0, \ldots, B_n\} \subset \bar{\mathbb{R}}^Y.$$

In $\bar{\mathbb{R}}^X$, we define the addition and the scalar multiplication operations respectively as follows:

$$(\forall f, g \in \bar{\mathbb{R}}^X)((\forall x \in X)((f+g)(x) \triangleq f(x)+g(x)));$$
$$(\forall f \in \bar{\mathbb{R}}^X)(\forall a \in \mathbb{R})((\forall x \in X)((af)(x) = af(x))).$$

It is easy to know that $(\bar{\mathbb{R}}^X, +, \cdot)$ and $(\bar{\mathbb{R}}^Y, +, \cdot)$ are linear spaces over the real number field $\mathbb{R}$, the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are defined respectively such that $(\bar{\mathbb{R}}^X, \|\cdot\|_X)$ and $(\bar{\mathbb{R}}^Y, \|\cdot\|_Y)$ all become the normed linear spaces. Thus $\mathcal{A}$ and $\mathcal{B}$ are able to be regarded as the subsets of the normed linear spaces $\bar{\mathbb{R}}^X$ and $\bar{\mathbb{R}}^Y$, respectively. So-called a group of fuzzy inference rules, $A_i \to B_i, i = 0, 1, \ldots, n$, is actually a transformation from a subset $\mathcal{A}$ of the normed linear space $\bar{\mathbb{R}}^X$ to a subset $\mathcal{B}$ of the normed linear space $\bar{\mathbb{R}}^Y$, denoted by $T_{n+1}$, i.e.

$$T_{n+1} : \mathcal{A} \longrightarrow \mathcal{B}, A_i \mapsto T_{n+1}(A_i) \triangleq B_i, i = 0, 1, \ldots, n. \tag{9}$$

Whether can this transformation be expanded as a transformation from the normed linear space $\bar{\mathbb{R}}^X$ to the normed linear space $\bar{\mathbb{R}}^Y$? If can, then the transformation is denoted by $T: \bar{\mathbb{R}}^X \to \bar{\mathbb{R}}^Y$. In fact, this thing has already almost been done (see (5)), i.e., $T = \circ$. However, $T$ is not a linear transformation but a nonlinear transformation. In other words, $T$ is not a bounded linear operator from the normed linear space $(\bar{\mathbb{R}}^X, \|\cdot\|_X)$ to the normed linear space $(\bar{\mathbb{R}}^Y, \|\cdot\|_Y)$. Moreover, the transformation as (5) has a bad property, that is, $T|_{\mathcal{A}} \neq T_{n+1}$, or $T$ does not hold that $T(A_i) = B_i, i = 0, 1, \ldots, n$. But this does not retard us to use the norms. Although $T(A_i) \neq B_i$, $\|T(A_i) - B_i\|_Y$ does describe degree for $T(A_i)$ approximating $B_i$, which means the transformation as (5) is not an interpolation operator but a fitted operator in accordance with (9). We have denoted $X = [a, b]$ and $Y = [c, d]$ that represent the input-output connection of the uncertain system to be a function, as $s : [a, b] \to [c, d]$, $x \mapsto s(x)$. In fact, $s(x)$ is usually a continuous function, i.e., $s \in C[a, b]$. So we take account of our problems in $C[a, b]$. Firstly, $\forall s \in C[a, b]$, a norm in $C[a, b]$ is defined as follows: $\|s\|_\infty \triangleq \max\{|s(x)||x \in [a, b]\}$.

Reviewing the group of fuzzy inference rules as (1), when $A_i$ and $B_i$ are all normal fuzzy sets which mean $\exists x_i \in X$ and $\exists y_i \in Y$ such that $A(x_i) = 1$ and $B(y_i) = 1$, this shows us to know that, designing a group of fuzzy inference rules as (1) and obtaining a group of input-output data $\{(x_i, y_i)|i = 0, 1, \ldots, n\}$ is almost a same thing, where this group of data should satisfy the following interpolation condition: $y_i = s(x_i), i = 0, 1, \ldots, n$. Theorem 1 means $s(x) \approx \underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^n A_i(x)y_i = \sum_{i=0}^n A_i(x)s(x_i)$, and we are able to get a bounded linear operator from $(C[a, b], ||\cdot||_\infty)$ to $(C[a, b], ||\cdot||_\infty)$ as follows:

$$L_{n+1} : C[a, b] \longrightarrow C[a, b], s \mapsto L_{n+1}(s), L_{n+1}(s)(x) \triangleq F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)s(x_i). \tag{10}$$

Based on this operator, a sort of numerical characters of the fuzzy systems shown in Theorem 1 can be described by the normal number of the operator as $||L_{n+1}||_\infty$, called by us normal numbers of fuzzy systems.

**Definition 3.** Let $L_{n+1}$ be a bounded linear operator from $C[a, b], ||\cdot||_\infty$ to $C[a, b], ||\cdot||_\infty$ based on some fuzzy system $s(x)$ defined as (10). Then $||L_{n+1}||_\infty$ is called the normal number of fuzzy system $s(x)$. Especially,

1) If $||L_{n+1}||_\infty = 1$, then $s(x)$ is called normal fuzzy system.

2) If there exists $M > 0$ such that $\sup\{||L_{n+1}||_\infty | n = 0, 1, 2, \ldots\} < M$, then $s(x)$ is called regular fuzzy system.

3) If $\lim_{n\to+\infty} ||L_{n+1}||_\infty = +\infty$, then $s(x)$ is called sigular fuzzy system.

Clearly, we have the following conclusion.

**Theorem 2.** If the bounded fuzzy sets as the antecedents of fuzzy inference rules are $A_i \in C[a, b]$, $i = 0, 1, \ldots, n$, then the normal number of the fuzzy system is the following

$$||L_{n+1}|| = \max_{x\in[a,b]} \sum_{i=0}^{n} |A_i(x)|. \tag{11}$$

This means $\|L_{n+1}\| = \max_{x\in[a,b]} \sum_{i=0}^{n} |A_i(x)|$ is a center invariant, which its quantity is sometimes only depending on the partitions on the input universe $X$.

**Example 2.** When the bounded fuzzy sets $A_i$ $(i = 0, 1, \ldots, n)$ as the antecedents of fuzzy inference rules are taken as the triangle wave fuzzy sets based on the partition of $X : a = x_0 < x_1 < \cdots < x_n = b$. The normal number of the fuzzy system has a unit quantity: $\sum_{i=0}^{n} A_i(x) \equiv 1$. We also have $\underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)y_i$. Clearly, $\{A_i(x)|i = 0, 1, \ldots, n\}$ satisfies Kronecker properties, so $F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)y_i$ is a piecewise interpolation function based on basis functions $\{A_i(x)|i = 0, 1, \ldots, n\}$, which is called Piecewise linear fuzzy system.

**Example 3.** When the bounded fuzzy sets $A_i$ $(i = 0, 1, \ldots, n)$ as the antecedents of fuzzy inference rules are taken as Bernstein polynomial basis functions as follows:

$$A_i(x) = C_n^i \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{(n-i)}, i = 0, 1, \ldots, n, \tag{12}$$

then $\sum_{i=0}^{n} A_i(x) = 1$, and we have the fuzzy system

$$\underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)y_i = \sum_{i=0}^{n} C_n^i \left(\frac{x-a}{b-a}\right)^i \left(\frac{b-x}{b-a}\right)^{(n-i)} y_i, \tag{13}$$

which are called Bernstein fuzzy system. Especially, when $a = 0, b = 1$, $A_i(x) = C_n^i x^i(1-x)^{n-i}$, which is typical Bernstein polynomial function. But these basis functions $A_i(x) = C_n^i x^i(1-x)^{n-i}$ do not satisfy Kronecker properties. It is easy to verify that $F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)y_i$ is not interpolation function, but quasi-interpolation function.

**Example 4.** When the bounded fuzzy sets $A_i$ $(i = 0, 1, \ldots, n)$ as the antecedents of fuzzy inference rules are taken as Lagrange basis functions based on the partition of $X : a = x_0 < x_1 < \cdots < x_n = b$ as

$$A_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x-x_j}{x_i-x_j} = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} (i = 0, 1, \ldots, n). \tag{14}$$

It is easy to obtain $\sum_{i=0}^{n} A_i(x) = \sum_{i=0}^{n} \prod_{j=0,j\neq i}^{n} \dfrac{x-x_j}{x_i-x_j} \equiv 1$. We also have that

$$\underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)y_i = \sum_{i=0}^{n} \prod_{j=0,j\neq i}^{n} \frac{x-x_j}{x_i-x_j} y_i. \tag{15}$$

$F_{n+1}(x) = \sum_{i=0}^{n} \prod_{j=0,j\neq i}^{n} \dfrac{x-x_j}{x_i-x_j} y_i$ is interpolation function because basis functions $\{A_i|i = 0, 1, \ldots, n\}$ are typical Lagrange interpolation basis functions and satisfy the Kronecker properties. This kind of fuzzy systems is called Lagrange fuzzy system.

**Corollary 1.** The normal numbers of piecewise linear fuzzy system and Bernstein fuzzy system have a unit quantity: $||L_{n+1}|| = 1$. So, the piecewise linear fuzzy systems and Bernstein fuzzy systems are normal fuzzy systems.

**Corollary 2.** The following inequality holds for Lagrange fuzzy systems (see [33–35]):

$$||L_{n+1}|| = \max_{x\in[a,b]} \sum_{i=0}^{n} |A_i(x)| > \frac{\ln(n+1)}{8\sqrt{\pi}}. \tag{16}$$

Lagrange fuzzy systems are singular fuzzy systems since $\lim_{n\to\infty} ||L_{n+1}|| = +\infty$.

**Note 1.** For the bounded linear operator $L_{n+1} \in C[a,b]^{C[a,b]}$, if it holds the following condition: $(\forall s \in C[a,b])(\exists M_s > 0)(\sup\{\|L_{n+1}(s)\|_\infty | n = 0, 1, \ldots\} < M_s)$, then the linear operator $L_{n+1}$ is called pointwise bounded. Because $C[a,b]$ is a Banach space, by utilizing Resonance Theorem in functional analysis (see [33, 34]): if a bounded linear operator $L_{n+1}$ is pointwise bounded, then $L_{n+1}$ must be uniformly bounded. Thus, the bounded linear operator $L_{n+1}$ in Lagrange fuzzy systems is not point-wise bounded, which means $(\exists s \in C[a,b])(\sup\{\|L_{n+1}(s)\|_\infty | n = 0,1,\ldots\} = +\infty)$. In other words, there exits a function $s(x)$ in $C[a,b]$ such that $L_{n+1}(s)(x)$ does not converge to $s(x)$. So when using Lagrange fuzzy systems, we should firstly verify their convergence.

Actually, Runge had found the flaw as "singular" on Lagrange fuzzy systems (see [28, 35]) and he gave a counterexample: considering a continuous function $s(x) = 1/(1+25x^2)$ in $C[-1,1]$, making an equidistant partition on $[-1,1]$ as $x_i = -1 + (2i/n), i = 0,1,\ldots,n$; straightway from numerical values we are able to learn that $L_{n+1}(s)(x)$ does not converge to $s(x)$. For example, when $n = 10$, it is not difficult to calculate that $L_{n+1}(s)(-0.96) = L_{11}(s)(-0.96) \approx 1.80438, s(-0.96) \approx 0.04160$. We are able to see how big the difference between them in a neighborhood at an endpoint. Similarly, the difference between $L_{11}(s)(0.96)$ and $s(0.96)$ is also big, in a neighborhood at another endpoint. If the counterexample given by Runge showed us a kind of no convergence of the interpolation function only at endpoints, then ones must ask such a question: whether can we find a continuous function $s \in C[a,b]$ such that $L_{n+1}(s)(x)$ does not converge to $s(x)$ a.e. in $[a,b]$ ? In fact, Bernstein for this thing had given a theorem in 1916 (see [35]): for the continuous function $s(x) = |x| \in C[-1,1]$, by making an equidistant partition on $[-1,1]$ as $x_i = -1 + (2i/n)$, $i = 0,1,\ldots,n$, $L_{n+1}(s)(x)$ does not converge to $s(x) = |x|$ at any point in $[-1,1]$ except the three points: $x = -1, 0, 1$, with $n \to \infty$.

**Note 2.** It is easy to know that $\{A_0, A_1, \ldots, A_n\}$ from Example 2 to Example 4, is a group of $n+1$ linearly independent functions in $C[a,b]$, and then an $n+1$ dimensional linear subspace of $C[a,b]$, $\text{span}\{A_0, A_1, \ldots, A_n\} \subset C[a,b]$, can be generated by $\{A_0, A_1, \ldots, A_n\}$ as basis functions of the linear subspace. This means, for an uncertain system, $s : X \to Y$, where $X = [a,b] \subset \mathbb{R}$, $Y = [c,d] \subset \mathbb{R}$, $s(x)$ is an undetermined input-output connection in which $s(x)$ is requested to be a continuous function, i.e., $s \in C[a,b]$. However, we only know a group of the input-output experiment data as $\{(x_i, y_i) | i = 0,1,\ldots,n\}$ holding interpolation condition: $y_i = s(x_i), i = 0,1,\ldots,n$. From this group of basic data we can generate a group of fuzzy inference rules: $A_i \longrightarrow B_i, i = 0,1,\ldots n$, where $A_i (i = 0,1,\ldots,n)$ are able to be the triangle wave fuzzy sets required of holding the double-phase property as $(\forall x \in X)(\exists i \in \{0,1,\ldots,n-1\})(A_i(x) + A_{i+1}(x) = 1)$ or Lagrange basis functions or Bernstein base functions, which are the bounded fuzzy sets and just forming the basis functions of $\text{span}\{A_0, A_1, \ldots, A_n\}$. By function approximation theory (see [27, 28]), we know that, arbitrarily given $s \in C[a,b]$, for any $\varepsilon > 0$, there exists a group of fuzzy inference rules, that is equivalent to a group of input-output data, $\{(x_i, y_i) | i = 0,1,\ldots,n\}$, such that $\|L_{n+1}(s) - s\|_\infty < \varepsilon$. By noticing that $L_{n+1}(s) \in \text{span}\{A_0, A_1, \ldots, A_n\}$, we know that, for any function $f(x)$ in the infinite dimensional function space $C[a,b]$, we are able to use a function $g(x)$ in the finite dimensional function space $\text{span}\{A_0, A_1, \ldots, A_n\}$ to approximate it. This makes a part of fuzzy system analysis be able to be framed into the approximation theory of functions.

### 3.2 Hermite fuzzy systems and their normal numbers

For improving differentiability of the fuzzy systems obtained by us, in this section, we take account of a new kind of fuzzy inference structure and the fuzzy systems generated by them. Now we turn to concern with a sort of single input double outputs open loop uncertain systems, where the input universe is that $X = [a,b] \subset \mathbb{R}$ and the output universe is that $Y \times Y' = [c,d] \times [c',d'] \subset \mathbb{R} \times \mathbb{R}$. Clearly, the double outputs variables $y$ and $y'$ are not independent, in which first output is that $y = s(x)$ and second output is that $y' = s'(x)$ generated in essence by first output.

Suppose a group of the input-output data have been got by us by use of experiments as $\{(x_i, (y_i, y'_i)) | i = 0,1,\ldots,n\}$, where $\{x_i | i = 0,1,\ldots,n\}$, the input data, can form a partition on the input universe $X = [a,b] \subset \mathbb{R}$ as $a = x_0 < x_1 < \ldots < x_n = b$, and $\{(y_i, y'_i) | i = 0,1,\ldots,n\}$, the output data, is a group of 2-dimensional vectors holding the following conditions:

1) The interpolation condition: $s(x_i) = y_i, s'(x_i) = \frac{\mathrm{d}s}{\mathrm{d}x}|_{x=x_i} = y'_i, i = 0,1,\ldots,n$.

2) The partition condition: $c = y_{k_0} < y_{k_1} < \cdots < y_{k_n} = d$ and $c' = y'_{p_0} < y'_{p_1} < \cdots < y'_{p_n} = d'$, where

$k_i = \sigma(i)$, $p_i = \tau(i)$, $i = 0, 1, \cdots, n$ and $\sigma$ and $\tau$ are two $(n+1)$-ary permutations:

$$\sigma = \left( \begin{array}{cccc} 0 & 1 & \cdots & n \\ k_0 & k_1 & \cdots & k_n \end{array} \right), \ \tau = \left( \begin{array}{cccc} 0 & 1 & \cdots & n \\ p_0 & p_1 & \cdots & p_n \end{array} \right).$$

We take account of how to construct the fuzzy inference rules for the uncertain system. Let

$$\Delta y_{k_i} = y_{k_{i+1}} - y_{k_i}, i = 0, 1, \ldots, n-1, \quad \Delta y_{k_n} = (\textstyle\sum_{i=0}^{n-1} \Delta y_{k_i})/n;$$
$$\Delta y'_{p_i} = y'_{p_{i+1}} - y'_{p_i}, i = 0, 1, \ldots, n-1, \quad \Delta y'_{p_n} = (\textstyle\sum_{i=0}^{n-1} \Delta y'_{p_i})/n.$$

Firstly, the two groups of $\bar{\mathbb{R}}$-fuzzy sets $B_i \in \bar{\mathbb{R}}^Y (i = 0, 1, \ldots, n)$ and $\hat{B}_i \in \bar{\mathbb{R}}^{Y'} (i = 0, 1, \ldots, n)$ as the consequents of the fuzzy inference rules as follows:

$$B_i(y) = \frac{\bar{B}_i(y)}{(\Delta y_i)^2 \sum_{j=0}^{n} \Delta y'_j}, \quad \hat{B}_i(y') = \frac{B'_i(y')}{(\Delta y'_i)^2 \sum_{j=0}^{n} \Delta y_j}, \quad i = 0, 1, \ldots, n, \quad y \in Y, \tag{17}$$

where the structures of the Zadeh fuzzy sets $B'_{k_i} \in \mathcal{F}(Y')$ fully resemble the ones of $\bar{B}_{k_i} \in \mathcal{F}(Y)$.

And then, we make the two groups of the bounded fuzzy sets $A_i \in BF(X)(i = 0, 1, \ldots, n)$ and $\hat{A}_i \in BF(X)(i = 0, 1, \ldots, n)$ as the antecedents of the fuzzy inference rules as the following:

$$A_0(x) = \begin{cases} \left(1 + 2\dfrac{x - x_0}{x_1 - x_0}\right)\left(\dfrac{x - x_1}{x_0 - x_1}\right)^2, & x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{A}_0(x) = \begin{cases} (x - x_0)\left(\dfrac{x - x_1}{x_0 - x_1}\right)^2, & x \in [x_0, x_1], \\ 0, & \text{otherwise}, \end{cases}$$

$$A_i(x) = \begin{cases} \left(1 + 2\dfrac{x - x_i}{x_{i-1} - x_i}\right)\left(\dfrac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2, & x \in [x_{i-1}, x_i), \\ \left(1 + 2\dfrac{x - x_i}{x_{i+1} - x_i}\right)\left(\dfrac{x_{i+1} - x}{x_{i+1} - x_i}\right)^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases}$$

$i = 1, 2, \ldots, n-1,$

$$\hat{A}_i(x) = \begin{cases} (x - x_i)\left(\dfrac{x - x_{i-1}}{x_i - x_{i-1}}\right)^2, & x \in [x_{i-1}, x_i), \\ (x - x_i)\left(\dfrac{x_{i+1} - x}{x_{i+1} - x_i}\right)^2, & x \in [x_i, x_{i+1}], \\ 0, & \text{otherwise}, \end{cases}$$

$i = 1, 2, \cdots, n-1$

$$A_n(x) = \begin{cases} \left(1 + 2\dfrac{x_n - x}{x_n - x_{n-1}}\right)\left(\dfrac{x - x_{n-1}}{x_n - x_{n-1}}\right)^2, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{A}_n(x) = \begin{cases} (x - x_n)\left(\dfrac{x - x_{n-1}}{x_n - x_{n-1}}\right)^2, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}. \end{cases}$$

Let $\mathcal{A} \triangleq \{A_i | i = 0, 1, \ldots, n\}$, $\hat{\mathcal{A}} \triangleq \{\hat{A}_i | i = 0, 1, \ldots, n\}$, $\mathcal{B} \triangleq \{B_i | i = 0, 1, \ldots, n\}$ and $\hat{\mathcal{B}} \triangleq \{\hat{B}_i | i = 0, 1, \ldots, n\}$. It is easy to verify that the elements in $\mathcal{A}$, $\hat{\mathcal{A}}$, $\mathcal{B}$ and $\hat{\mathcal{B}}$ satisfy the following conditions:

1) $A_i (i = 0, 1, \ldots, n)$ and $\hat{A}_i (i = 0, 1, \ldots, n)$ hold Hermite interpolation condition:

$$A_i(x_j) = \delta_{ij}, \quad \frac{\mathrm{d}A_i(x)}{x}|_{x=x_j} = 0; i, j = 0, 1, \ldots, n;$$
$$\hat{A}_i(x_j) = 0, \quad \frac{\mathrm{d}\hat{A}_i(x)}{x}|_{x=x_j} = \delta_{ij}; i, j = 0, 1, \ldots, n;$$

2) $A_i (i = 0, 1, \cdots, n)$ meet the normalizing condition: $\sum_{i=0}^n A_i(x) \equiv 1$ and

$$A_i(x)A_j(x) \neq 0, \text{ for } |j-i| \leqslant 1, \quad i,j = 0,1,\ldots,n,$$
$$A_i(x)A_j(x) = 0, \text{ for } |j-i| \geqslant 2, \quad i,j = 0,1,\ldots,n.$$

By use of $\mathcal{A}$, $\hat{\mathcal{A}}$, $\mathcal{B}$ and $\hat{\mathcal{B}}$, a group of 2-dimensional vector fuzzy inference rules is able to be formed as follows:

$$(\mathcal{A}, \hat{\mathcal{A}}) \longrightarrow (\mathcal{B}, \hat{\mathcal{B}}) : (A_i, \hat{A}_i) \longrightarrow (B_i, \hat{B}_i), \ i = 0,1,\ldots,n. \tag{18}$$

According to (18), the whole $\bar{R}$-fuzzy inference relation $R$ should be with the following form:

$$R(x,(y,y')) = \sum_{i=0}^{n}(w_{1i}A_i(x)B_i(y) + w_{2i}\hat{A}_i(x)\hat{B}_i(y')), \tag{19}$$

$$w_{1i} \triangleq \Delta y_i / \sum_{j=0}^{n} \Delta y_j, w_{2i} \triangleq \Delta y'_i / \sum_{j=0}^{n} \Delta y'_j, i = 0,1,\ldots,n. \tag{20}$$

Clearly, $(\forall i \in \{0,1,\ldots,n\})(\forall k \in \{1,2\})(w_{ki} > 0)$, $(\forall k \in \{1,2\})(\sum_{i=0}^n w_{ki} = 1)$.

**Theorem 3.** Reserving the notations and the notions defined above, based on the 2-dimensional vector fuzzy inference rules as (18), by means of CRI method, a fuzzy system $\underline{s}$ obtained by us is approximately equal to a Hermite interpolation function which basis functions of it are just the bounded fuzzy sets $A_i(x)$ and $\hat{A}_i(x)$, $i = 0,1,\ldots,n$, i.e.,

$$\underline{s}(x) \approx F_{n+1}(x) \triangleq \sum_{i=0}^{n}(A_i(x)y_i + \hat{A}_i(x)y'_i). \tag{21}$$

*Proof.* By CRI method, from the $\bar{\mathbb{R}}$-fuzzy inference relation (see (19)), an $\bar{\mathbb{R}}$-fuzzy transformation $\circ$ is induced as follows:

$$\circ : \bar{\mathbb{R}}^X \to \bar{\mathbb{R}}^Y \times \bar{\mathbb{R}}^{Y'}, \ A \mapsto B \triangleq \circ(A) \triangleq A \circ R.$$

For any $x^* \in X$, let $A^*(x) \triangleq \begin{cases} M & x = x^*; \\ m & x \neq x^*, \end{cases}$ where

$$M \triangleq \sup\{R(x,(y,y'))|(x,(y,y')) \in [a,b] \times ([c,d] \times [c',d'])\},$$
$$m \triangleq \inf\{R(x,(y,y'))|(x,(y,y')) \in [a,b] \times ([c,d] \times [c',d'])\}.$$

A fuzzy inference result $B^* \in \bar{\mathbb{R}}^Y \times \bar{\mathbb{R}}^{Y'}$ from (5) is got as the following $B^*(y,y') = B_1^*(y) + B_2^*(y')$, where $B_1^*(y) \triangleq \sum_{i=0}^n w_{1i}A_i(x^*)B_i(y)$ and $B_2^*(y') \triangleq \sum_{i=0}^n w_{2i}\hat{A}_i(x^*)\hat{B}_i(y')$.

Now we take account of how to change the binary $\bar{\mathbb{R}}$-fuzzy set $B^*(y,y')$ into a corresponding point $y^* \in Y$ for $x^* \in X$. If $\int_Y |yB_1^*(y)|dy < \infty$, $\int_{Y'} |y'B_2^*(y')|dy' < \infty$, $\int_Y |B_1^*(y)|dy < \infty$, $\int_{Y'} |B_2^*(y')|dy' < \infty$, $\int_Y B_1^*(y)dy \neq 0$ and $\int_{Y'} B_2^*(y')dy' \neq 0$, then let

$$y_1^* \triangleq \frac{\int_Y yB_1^*(y)dy}{\int_Y B_1^*(y)dy}, \quad y_2^* \triangleq \frac{\int_{Y'} y'B_2^*(y')dy'}{\int_{Y'} B_2^*(y')dy'}.$$

We will use the weighted sum to determine such $y^*$, i.e., $y^* = \lambda_1 y_1^* + \lambda_2 y_2^*$, where

$$\lambda_1 \triangleq \frac{\int_Y B_1^*(y)dy}{\int_Y B_1^*(y)dy + \int_{Y'} B_2^*(y')dy'}, \quad \lambda_2 \triangleq \frac{\int_{Y'} B_2^*(y')dy'}{\int_Y B_1^*(y)dy + \int_{Y'} B_2^*(y')dy'}.$$

Then, $\lambda_1 \geqslant 0, \lambda_2 \geqslant 0$ and $\lambda_1 + \lambda_2 = 1$. Let $y^* = \lambda_1 y_1^* + \lambda_2 y_2^*$. Thus we have

$$y^* = \lambda_1 y_1^* + \lambda_2 y_2^* = \frac{\int_Y y(\sum_{i=0}^n w_{1i}A_i(x^*)B_i(y))dy + \int_{Y'} y'(\sum_{i=0}^n w_{2i}\hat{A}_i(x^*)\hat{B}_i(y'))dy'}{\int_Y \sum_{i=0}^n w_{1i}A_i(x^*)B_i(y)dy + \int_{Y'} \sum_{i=0}^n w_{2i}\hat{A}_i(x^*)\hat{B}_i(y')dy'}. \tag{22}$$

Since $x^*$ is arbitrary in $X$, $(x^*, y^*)$ can be generalized to be $(x, y)$, and $\underline{s}(x)$ is replaced by $y$. By noticing that $\sum_{i=0}^n A_i(x) \equiv 1$ and that the definite integrals are replaced by Riemann sums, we have

$$\begin{aligned}\underline{s}(x) &= \frac{\int_Y y(\sum_{i=0}^n w_{1i}A_i(x)B_i(y))dy + \int_{Y'} y'(\sum_{i=0}^n w_{2i}\hat{A}_i(x)\hat{B}_i(y'))dy'}{\int_Y \sum_{i=0}^n w_{1i}A_i(x)B_i(y)dy + \int_{Y'} \sum_{i=0}^n w_{2i}\hat{A}_i(x)\hat{B}_i(y')dy'} \\ &\approx \frac{\sum_{i=0}^n \frac{\Delta y_i A_i(x)}{\sum_{j=0}^n \Delta y_j}\cdot\frac{y_i\bar{B}_i(y_i)}{(\Delta y_i)^2\sum_{j=0}^n \Delta y_j'}\Delta y_i + \sum_{i=0}^n \frac{\Delta y_i'\hat{A}_i(x)}{\sum_{j=0}^n \Delta y_j'}\cdot\frac{y_i'B_i'(y_i')}{(\Delta y_i')^2\sum_{j=0}^n \Delta y_j}\Delta y'}{\sum_{i=0}^n \frac{\Delta y_i A_i(x)}{\sum_{j=0}^n \Delta y_j}\cdot\frac{\bar{B}_i(y_i)}{(\Delta y_i)^2\sum_{j=0}^n \Delta y_j'}\Delta y_i + \sum_{i=0}^n \frac{\Delta y_i'\hat{A}_i(x)}{\sum_{j=0}^n \Delta y_j'}\cdot\frac{B_i'(y_i')}{(\Delta y_i')^2\sum_{j=0}^n \Delta y_j}\Delta y'} \\ &= \frac{\sum_{i=0}^n A_i(x)y_i + \sum_{i=0}^n \hat{A}_i(x)y'}{\lambda_{n+1}(x)} = \frac{F_{n+1}(x)}{\lambda_{n+1}(x)},\end{aligned}$$

where we have put $\alpha_{n+1}(x) \triangleq \sum_{i=0}^n \hat{A}_i(x)$, $\lambda_{n+1}(x) \triangleq 1+\alpha_{n+1}(x)$ and $F_{n+1}(x) \triangleq \sum_{i=0}^n A_i(x)y_i + \sum_{i=0}^n \hat{A}_i(x)y'$. Or we have the following form:

$$\lambda_{n+1}(x)\underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^n A_i(x)y_i + \sum_{i=0}^n \hat{A}_i(x)y'. \tag{23}$$

Now we prove that $\lim_{n\to\infty} \lambda_{n+1}(x) = 1$, and this only needs to prove that $\lim_{n\to\infty} \alpha_{n+1}(x) = 0$. For doing this thing, we introduce a new notion for the universe partitions. In fact, let $h_i \triangleq x_{i+1} - x_i, i = 0, 1, \ldots, n-1; d_n \triangleq \max\{h_i | i = 0, 1, \ldots, n-1\}$. A partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ on the universe $X = [a, b]$ is called Conforming, if it meets the condition: $\lim_{n\to\infty} d_n = 0$.

Actually, the conforming for the partitions on universes is really reasonable, so that we assume that the partitions doing by us on the input universe $X$ are always conforming. Now we put that $x_{i+\frac{1}{2}} \triangleq (x_i + x_{i+1})/2$ and take an $x \in X$ arbitrarily. Clearly, $\exists i \in \{0, 1, \ldots, n-1\}$ such that $x \in [x_i, x_{i+1}]$, and it is easy to know that

$$\alpha_{n+1}(x) = \sum_{i=0}^n \hat{A}_i(x) = -2\frac{(x-x_i)(x_{i+1}-x)(x-x_{i+\frac{1}{2}})}{(x_{i+1}-x_i)^2}.$$

By using the elementary inequality: $\prod_{k=1}^p a_k \leqslant ((\sum_{k=1}^p a_k)/p)^p$, $a_k \geqslant 0$, $k = 1, 2, \ldots, p$, $p \geqslant 1$, when $x \in [x_i, x_{i+\frac{1}{2}}]$, $\alpha_{n+1}(x)$ has the following estimating expression:

$$\alpha_{n+1}(x) < \frac{1}{(x_{i+1}-x_i)^2}\cdot\left(\frac{1}{3}(2x - 2x_i + x_{i+1} - x + x_{i+\frac{1}{2}} - x)\right)^3 = \frac{h_i}{8}.$$

And when $x \in [x_{i+\frac{1}{2}}, x_{i+1}]$, $-\alpha_{n+1}(x)$ is of the following estimating expression:

$$-\alpha_{n+1}(x) < \frac{1}{(x_{i+1}-x_i)^2}\cdot\left(\frac{1}{3}(x - x_i + 2x_{i+1} - 2x + x - x_{i+\frac{1}{2}})\right)^3 = \frac{h_i}{8}.$$

From the above reasoning and the partition on $X$ being conforming, we have

$$(\forall x \in X)(|\alpha_{n+1}(x)| \leqslant \frac{h_i}{8} \leqslant \frac{d_n}{8} \to 0(n\to\infty)).$$

This is $(\forall x \in X)(\alpha_{n+1}(x) \to 0(n\to\infty))$ or $(\forall x \in X)(\lambda_{n+1}(x) \to 1(n\to\infty))$. Therefore, (23) should become that

$$\underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^n A_i(x)y_i + \sum_{i=0}^n \hat{A}_i(x)y'. \tag{24}$$

It is well-known that $F_{n+1}(x) = \sum_{i=0}^n A_i(x)y_i + \sum_{i=0}^n \hat{A}_i(x)y'$ is just a Hermite interpolation function, where not only $F_{n+1}(x)$ converges uniformly to $\underline{s}(x)$ but also $\frac{\mathrm{d}F_{n+1}(x)}{\mathrm{d}x}$ converges uniformly to $\frac{\mathrm{d}\underline{s}(x)}{\mathrm{d}x}$. Here such $\underline{s}$ is called a Hermite fuzzy system.

**Note 3.** Reviewing eq. (23), $\lambda_{n+1}(x)\underline{s}(x) \approx F_{n+1}(x) = \sum_{i=0}^{n} A_i(x)y_i + \sum_{i=0}^{n} \hat{A}_i(x)y'$, which makes us discover a phenomena: for a fuzzy system $\underline{s}$, although its input-output function $\underline{s}(x)$ is often not approximately equal to a typical interpolation function, as if the integrating factor method in differential equations, there may exist a non-zero function $\lambda_{n+1}(x)$ such that $\lambda_{n+1}(x)\underline{s}(x)$ is just approximately equal to a typical interpolation function. The fuzzy system $\underline{s}$ is called a Collocation factor fuzzy system. Under this significance, Hermite fuzzy systems are a kind of particular collocation factor fuzzy systems.

**Note 4.** Now we rewrite the typical triangle wave Zadeh fuzzy sets in Example 2 to be $\bar{A}_i$ $(i = 0, 1, \cdots, n)$, then $A_i$ and $\hat{A}_i$ will be represented as

$$\begin{aligned}
&A_i(x) = 3(\bar{A}_i(x))^2 - 2(\bar{A}_i(x))^3,\ x \in [a, b],\ i = 0, 1, \ldots, n;\\
&\hat{A}_0(x) = h_0\bar{A}_1(x)(\bar{A}_0(x))^2 x \in [a, b],\\
&\hat{A}_i(x) = \begin{cases} -h_{i-1}\bar{A}_{i-1}(x)(\bar{A}_i(x))^2, & x \in [a, x_i);\\ h_i\bar{A}_{i+1}(x)(\bar{A}_i(x))^2, & x \in [x_i, b]; \end{cases}\\
&\qquad\quad i = 1, 2, \ldots, n-1,\\
&\hat{A}_n(x) = -h_{n-1}\bar{A}_{n-1}(x)(\bar{A}_n(x))^2,\ x \in [a, b].
\end{aligned}$$

This means, in many cases, the triangle wave Zadeh fuzzy sets are the kernels of bounded fuzzy sets, in which the kernels play roles in generating these bounded fuzzy sets.

It is the time to calculate the normal numbers of Hermite fuzzy systems. At first, the norm in $(C^1[a, b], ||\cdot||_{C^1})$ is defined as $||s||_{C^1} \triangleq \max\{|s(x)|, |s'(x)||x \in [a, b]\}$. Secondly, we define a bounded linear operator from $(C^1[a, b], ||\cdot||_{C^1})$ to $(C^1[a, b], ||\cdot||_\infty)$ as follows:

$$L_{n+1} : C^1[a, b] \to C^1[a, b], s \mapsto L_{n+1}(s),$$

$$L_{n+1}(s)(x) \triangleq \sum_{i=0}^{n} A_i(x)y_i + \sum_{i=0}^{n} \hat{A}_i(x)y_i' = \sum_{i=0}^{n}(A_i(x)s(x_i) + \hat{A}_i(x)s'(x_i)), \tag{25}$$

where $s(x_i) = y_i$ and $s'(x_i) = \dfrac{\mathrm{d}s(x)}{\mathrm{d}x}|_{x=x_i} = y_i'$.

**Theorem 4.** Still by denoting $d_n \triangleq \max\{h_i|i = 0, 1, \ldots, n-1\}$, then the normal numbers of Hermite fuzzy systems are as the following: $||L_{n+1}|| = 1 + \frac{d_n}{4}$.

*Proof.* From $\sum_{i=0}^{n} A_i(x) = 1$, we have $||L_{n+1}(s)||_\infty = ||s||_{C^1}(1 + \max_{x\in[a,b]} \sum_{i=0}^{n} |\hat{A}_i(x)|)$. By the structures of the bounded fuzzy sets $\hat{A}(i = 0, 1, \ldots, n)$, for arbitrarily given $x \in X$, there exists an $i \in \{0, 1, \ldots, n-1\}$, such that $x \in [x_i, x_{i+1}]$. Thus

$$\sum_{k=0}^{n} |\hat{A}_k(x)| = \frac{(x_{i+1} - x)(x - x_i)}{x_{i+1} - x_i} \leqslant \frac{((x_{i+1} - x + x - x_i)/2)^2}{x_{i+1} - x_i} = \frac{h_i}{4} \leqslant \frac{d_n}{4}.$$

So $\|L_{n+1}\| = \sup\limits_{||s||_{C^1}\leqslant 1} ||L_{n+1}(s)||_\infty \leqslant 1 + d_n/4$. On the other hand, for arbitrarily given $\varepsilon \in (0, 0.1]$, we construct a function $\tilde{s}(x)$ as follows:

$$\tilde{s}(x) = \begin{cases} \exp[(1-\varepsilon)(x - x_{i_0})] - \varepsilon, & x \in [a, x_{i_0}];\\ 1 + (1 - 9\varepsilon)\left(\dfrac{x - x_{i_0+\frac{1}{2}}}{d_n}\right)^2 - (4 - 20\varepsilon)\left(\dfrac{x - x_{i_0+\frac{1}{2}}}{d_n}\right)^4, & x \in (x_{i_0}, x_{i_0+1});\\ \exp[-(1-\varepsilon)(x - x_{i_0+1})] - \varepsilon, & x \in [x_{i_0+1}, b], \end{cases}$$

where $i_0 \triangleq \min\{i \in \{0, 1, \ldots, n-1\}|h_i = d_n\}$ and $x_{i_0+\frac{1}{2}} = (x_{i_0} + x_{i_0+1})/2$. Clearly, $\tilde{s} \in C^1[a, b]$. At last, from the expressions of $\tilde{s}(x)$ and in $[a, b]$, it is easy to know that

$$\max_{x\in[a,x_{i_0}]}\{|s(x)|, |s'(x)|\} = \max_{x\in[a,x_{i_0}]}\{\exp[(1-\varepsilon)(x - x_{i_0})] - \varepsilon, (1-\varepsilon)\exp[(1-\varepsilon)(x - x_{i_0})]\} = 1 - \varepsilon,$$

$$\max_{x\in[x_{i_0+1},b]}\{|s(x)|,|s'(x)|\} = \max_{x\in[x_{i_0+1},b]}\{\exp[-(1-\varepsilon)(x-x_{i_0+1})]-\varepsilon,(1-\varepsilon)\exp[-(1-\varepsilon)(x-x_{i_0+1})]\}$$
$$= 1-\varepsilon.$$

And when $x \in (x_{i_0}, x_{i_0+1})$, it is also easy to know that

$$|s(x)| = 1 + (1-9\varepsilon)\left(\frac{x - x_{i_0+\frac{1}{2}}}{d_n}\right)^2 - (4-20\varepsilon)\left(\frac{x - x_{i_0+\frac{1}{2}}}{d_n}\right)^4,$$
$$|s'(x)| = \left|(2-18\varepsilon)\left(\frac{x - x_{i_0+\frac{1}{2}}}{d_n}\right) - (16-80\varepsilon)\left(\frac{x - x_{i_0+\frac{1}{2}}}{d_n}\right)^3\right|.$$

Thus $\max_{x\in(x_{i_0},x_{i_0+1})}\{|s(x)|,|s'(x)|\} = 1$. Based on the above three cases, we have that $||\tilde{s}||_{C^1} = 1$.

By noticing that $\tilde{s}(x_{i_0}) = \tilde{s}(x_{i_0+1}) = \tilde{s}'(x_{i_0}) = 1-\varepsilon$ and $\tilde{s}'(x_{i_0+1}) = \varepsilon - 1$, we have $||L_{n+1}|| = (1-\varepsilon)(1+\frac{d_n}{4})$.

Because $\varepsilon$ is arbitrary, it must be true that $||L_{n+1}|| \geqslant 1 + \frac{d_n}{4}$. Therefore, we have got that $||L_{n+1}|| = 1 + \frac{d_n}{4}$.

**Note 5.** We easily know that, when the partition on the universe $X$ is conforming, the sequence of numbers $d_n \to 0 (n \to \infty)$; so $d_n$ is bounded. Thus Hermite fuzzy systems are not the singular fuzzy systems, but the regular fuzzy systems. Although Hermite fuzzy systems are not the normal fuzzy systems, their limit systems are just the normal fuzzy systems.

## 4 Generalized Bernstein fuzzy system and its universal approximation property

We have introduced Bernstein fuzzy systems in Example 3, and now we begin to discuss universal approximation of such fuzzy systems. Here we let $X = [a, b] \subset \mathbb{R}$ and $Y = [c, d] \subset \mathbb{R}$, in which we are able to assume that $a = 0$ and $b = 1$, or else, we are able to make change of variable as $u = (x-a)/(b-a)$, then $u \in [0, 1]$. Therefore, we regarded the input-output connection of the system $s : X \to Y$ as $s \in C[0, 1]$. Assume that, for the information of $s(x)$, we only learn a group of the input-output data $\{(x_i, y_i)|i = 0, 1, \ldots, n\}$ holding the condition:

$$0 = x_0 < x_1 < \cdots < x_n = 1, \quad y_i = s(x_i), \quad i = 0, 1, \ldots, n.$$

The bounded fuzzy sets $A_i \in BF(X) (i = 0, 1, \ldots, n)$ as the antecedents of fuzzy inference are taken as $A_i(x) = C_n^i x^i (1-x)^{n-i}, i = 0, 1, \ldots, n$. Clearly $\{A_0, A_1, \ldots, A_n\}$ is a group of linearly independent functions in $C[0, 1]$ and easy to know it holding the normalizing condition: $\sum_{i=0}^n A_i(x) \equiv 1$. The bounded fuzzy sets $B_i (i = 0, 1, \ldots, n)$ as the consequents of fuzzy inference rules are still taken as that in section 2. Thus we get a group of fuzzy inference rules: $A_i \to B_i, i = 0, 1, \ldots, n$, then a fuzzy system $\underline{s}$ can be constructed to approximately be a generalized Bernstein polynomial as follows:

$$s(x) \approx \underline{s}(x) \approx \bar{B}_n(s; x) \triangleq \sum_{i=0}^n A_i(x) y_i = \sum_{i=0}^n A_i(x) s(x_i) = \sum_{i=0}^n s(x_i) C_n^i x^i (1-x)^{n-i}. \quad (26)$$

This is a particular example of Example 2. It is well-known that a Bernstein polynomial is a polynomial as follows:

$$B_n(s; x) \triangleq \sum_{i=0}^n s\left(\frac{i}{n}\right) C_n^i x^i (1-x)^{n-i}. \quad (27)$$

Be careful, in a Bernstein polynomial, the partition on $X = [0, 1]$ is equidistant as $x_i = i/n, i = 0, 1, \ldots, n$; but in a generalized Bernstein polynomial $\bar{B}_n(s; x)$, the partition on $X = [0, 1]$ is unnecessarily equidistant. Bernstein polynomial $B_n(s; x)$ had been ever used to prove Weierstrass First Approximation Theorem, which means $B_n(s; x)$ can uniformly converge to $s(x)$ on [0, 1], i.e., $\lim_{n\to\infty} \|s - B_n(s)\|_\infty = 0$. However, we have such a question: whether can a generalized Bernstein polynomial $\bar{B}_n(s; x)$ uniformly converge to

$s(x)$ on [0, 1], too? The answer should be not. Now we concern with under some condition to prove the proposition. Let $B[0,1]$ be the set of all bounded functions defined on [0, 1]. We firstly in $B[0,1]$ take account of the convergence of generalized Bernstein polynomials $\bar{B}_n(s;x)$.

**Theorem 5.** Let $s \in B[0,1]$ and that $x \in [0,1]$ be any continuous point of $s(x)$. For any given real number $\rho > 0$, if a partition on [0, 1], $0 = x_0 < x_1 < \cdots < x_n = 1$, satisfies the condition: $\max_i |x_i - \frac{i}{n}| \leqslant \frac{\rho}{n}$, then $n$th ($n \geqslant 1$) generalized Bernstein polynomial $\bar{B}_n(s;x)$ converges to $s(x)$, i.e., $\lim_{n\to\infty} \bar{B}_n(s;x) = s(x)$.

*Proof* We first prove a useful inequality: $\forall \delta > 0, \forall x \in [0,1]$, we have $\sum_{|x_i - x| \geqslant \delta} C_n^i x^i (1-x)^i \leqslant \frac{4\rho^2+n}{2n^2\delta}$, where $\Sigma_{|x_i-x|\geqslant\delta}$ means to do sum for all $i$ that hold $|x_i - x| \geqslant \delta$. In fact, as $|x_i - x| \geqslant \delta$, we know that $\frac{(x_i-x)^2}{\delta^2} \geqslant 1$. Thus

$$\begin{aligned}
\sum_{|x_i-x|\geqslant\delta} C_n^i x^i (1-x)^{n-i} &\leqslant \sum_{|x_i-x|\geqslant\delta} \frac{1}{\delta^2}(x_i - x)^2 C_n^i x^i (1-x)^{n-i} \\
&\leqslant \frac{2}{n^2\delta^2} \sum_{|x_i-x|\geqslant\delta} [(nx_i - i)^2 + (i - nx)^2] C_n^i x^i (1-x)^{n-i} \\
&= \frac{2}{n^2\delta^2} \sum_{|x_i-x|\geqslant\delta} (nx_i - i)^2 C_n^i x^i (1-x)^{n-i} \\
&\quad + \frac{2}{n^2\delta^2} \sum_{|x_i-x|\geqslant\delta} (i - nx)^2 C_n^i x^i (1-x)^{n-i}.
\end{aligned}$$

Let $\mathrm{I} \triangleq \frac{2}{n^2\delta^2} \sum_{|x_i-x|\geqslant\delta} (nx_i - i)^2 C_n^i x^i (1-x)^{n-i}$ and $\mathrm{II} \triangleq \frac{2}{n^2\delta^2} \sum_{|x_i-x|\geqslant\delta} (i - nx)^2 C_n^i x^i (1-x)^{n-i}$. And it is not difficult to prove the identity: $\sum_{i=0}^n (i - nx)^2 C_n^i x^i (1-x)^{n-i} \equiv nx(1-x)$. Now by using our condition: $\max_i |x_i - \frac{i}{n}| \leqslant \frac{\rho}{n}$, we know that $(nx_i - i)^2 \leqslant \rho^2$, and then

$$\begin{aligned}
&\mathrm{I} \leqslant \frac{2\rho^2}{n^2\delta^2} \sum_{|x_i-x|\geqslant\delta} C_n^i x^i (1-x)^{n-i} \leqslant \frac{2\rho^2}{n^2\delta^2}, \\
&\mathrm{II} \leqslant \frac{2}{n^2\delta^2} \sum_{i=0}^{n} (i - nx)^2 C_n^i x^i (1-x)^{n-i} \leqslant \frac{n}{2n^2\delta^2}, \\
&\mathrm{I} + \mathrm{II} \leqslant \frac{2\rho^2}{n^2\delta^2} + \frac{n}{2n^2\delta^2} = \frac{4\rho^2 + n}{2n^2\delta^2}.
\end{aligned}$$

Secondly, we will prove $\bar{B}_n(s;x)$ converges to $s(x)$. Actually, as that $s \in B[0,1]$ means $s$ being bounded on $[0,1]$, we have that $(\exists M > 0)(\forall t \in [0,1])(|s(t)| \leqslant M)$; and since $x$ is a continuous point of $s(x)$, so

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x' \in [0,1])(|x - x'| \leqslant \delta \Rightarrow |s(x) - s(x')| < \varepsilon/2).$$

Besides by $\lim_{n\to\infty} \frac{4\rho^2+n}{2n^2\delta^2} = 0$, for above $\varepsilon$, there exits an $N \in \mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers), such that $(\forall n \geqslant N)(\frac{4\rho^2+n}{2n^2\delta^2} < \frac{\varepsilon}{4M})$. Thus, $\forall n \geqslant N$, we have

$$\begin{aligned}
|s(x) - \bar{B}_n(s;x)| &\leqslant \sum_{|x_i-x|\geqslant\delta} |s(x) - s(x_i)| C_n^i x^i (1-x)^{n-i} \\
&\quad + \sum_{|x_i-x|<\delta} |s(x) - s(x_i)| C_n^i x^i (1-x)^{n-i} \\
&\leqslant 2M \sum_{|x_i-x|\geqslant\delta} C_n^i x^i (1-x)^{n-i} + \frac{\varepsilon}{2} \sum_{|x_i-x|<\delta} C_n^i x^i (1-x)^{n-i} \\
&< 2M \frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This means $\lim_{n\to\infty} \bar{B}_n(s;x) = s(x)$.

By Theorem 5, we have the following conclusion.

**Theorem 6.** Let $s \in C[0,1]$. For any real number $\rho > 0$, if a partition on $X = [0,1]$, $0 = x_0 < x_1 < \cdots < x_n = 1$, holds the condition: $\max_i |x_i - \frac{i}{n}| \leqslant \frac{\rho}{n}$, then $n$th $(n > 1)$ generalized Bernstein polynomial $\overline{B}_n(s;x)$ converges uniformly to $s(x)$, i.e.,

$$\lim_{n\to\infty} \|\bar{B}_n(s) - s\|_\infty = \lim_{n\to\infty} \max_{x\in[0,1]} |\bar{B}_n(s;x) - s(x)| = 0. \tag{28}$$

**Note 6.** We are able to give a counterexample to show that there at least exists a generalized Bernstein polynomial $\bar{B}_n(s;x)$ such that it does not converge to $s(x)$; in other words, for a generalized Bernstein polynomial $\bar{B}_n(s;x)$, there at least exists a $s \in C[0,1]$, by making a particular partition on $[0,1]$, $0 = x_0 < x_1 < \cdots < x_n = 1$, such that $\bar{B}_n(s;x)$ does not converge to $s(x)$ in $[0,1]$.

**Example 5.** For $n \geqslant 2$, choosing a partition on $[0,1]$, $0 = x_0 < x_1 < \cdots < x_n = 1$, where $x_i (i = 0, 1, \cdots, n)$ are defined as follows:

$$x_i = \begin{cases} \dfrac{i}{\ln n}, & 0 \leqslant i \leqslant [\ln n], \\ \dfrac{[\ln n]}{\ln n} + \dfrac{\ln n - [\ln n]}{\ln n} \cdot \dfrac{i - [\ln n]}{n - [\ln n]}, & [\ln n] + 1 \leqslant i \leqslant n. \end{cases}$$

Clearly, $\lim_{n\to\infty} \max_{1\leqslant i\leqslant n}(x_i - x_{i-1}) = \lim_{n\to\infty} \frac{1}{\ln n} = 0$. For any given bounded function $s(x)$ on $[0,1]$, it is supposed that $s(x)$ holds the weak Lipschitz condition:

$$(\exists L > 0)(\forall x \in [0,1])(|s(1) - s(x)| \leqslant L(1 - x)). \tag{29}$$

By noticing that $\lim_{x\to 0} x^x = 1$, we should stipulate that $0^0 = 1$. Then the generalized Bernstein polynomial $\bar{B}_n(s;x) = \sum_{i=0}^n s(x_i) C_n^i x^i (1-x)^{n-i}$ satisfies the bounded condition: $\bar{B}_n(s;0) = s(0)$ and $\bar{B}_n(s;1) = s(1)$. When $x \in (0,1)$, by denoting that $\hat{x} \triangleq \max\{x, 1-x\}$, we have

$$|s(1) - \bar{B}_n(s;x)| = \left| s(1)\sum_{i=0}^n C_n^i x^i (1-x)^{n-i} - \sum_{i=0}^n s(x_i) C_n^i x^i (1-x)^{n-i} \right| \leqslant L\hat{x}^n n^{\ln n + 1} + \frac{L}{\ln n}.$$

So $\lim_{n\to\infty} \hat{x}^n n^{\ln n+1} = 0$, and then we can easily have that $\lim_{n\to\infty} \bar{B}_n(s;x) = s(1)$.

## 5 Conclusions

The definition of $\bar{\mathbb{R}}$-fuzzy sets and normal number of fuzzy systems are proposed in this paper. Our main results are as follows.

1) Some fuzzy systems are constucted by suitably using several kinds of $\bar{\mathbb{R}}$-fuzzy sets as fuzzy inference antecedents, such as interpolation fuzzy systems, the piecewise linear fuzzy system, Lagrange fuzzy system, Bernstein fuzzy systems and Hermite fuzzy systems.

2) Under the significance of the normal numbers of fuzzy systems, all fuzzy systems are able to be classified as three classes such as the normal fuzzy systems, the regular fuzzy systems and the singular fuzzy systems. It was proved that piecewise linear fuzzy systems and Bernstein fuzzy systems are normal fuzzy systems; Hermit fuzzy systems are regular fuzzy systems; Lagrange fuzzy systems are singular fuzzy systems.

3) By constructing Bernstein fuzzy system, generalized Bernstein polynomials are obtained. It is proved that generalized Bernstein polynomials are uniformly convergent in $C[a,b]$ under a weaker condition, and it is pointed out that there exist generalized Bernstein polynomials to be not convergent in $C[a,b]$ by use of constructing a counterexample.

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