三维强化采油渗流耦合系统隐式迎风分数步差分方法的收敛性分析

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摘要 对强化 (化学) 采油数值模拟中的渗流耦合系统问题, 本文提出隐式迎风分数步差分格式, 利用变分形式、能量方法、差分算子乘积交换性理论、高阶差分算子的分解、微分方程先验估计的理论和技巧, 得到最佳阶 l^2 模误差估计. 该方法已成功应用到强化采油数值模拟的生产实践中, 并且得到了很好的数值模拟效果.

关键词 强化 (化学) 采油 三维渗流耦合系统 隐式迎风分数步差分 最佳阶 l^2 估计 能源数值模拟

MSC (2010) 主题分类 65M06, 65N06, 76M25

1 引言

目前国内外行之有效的保持油藏压力的方法是注水开发, 其采收率比靠天然能量的开采方式高. 我国大庆油田在注水开发上取得了巨大的成绩, 使油田达到高产稳产. 如何进一步提高注水油藏的原油采收率, 仍然是一个具有战略性的重大课题^[1].

油田经水驱开采后, 油藏中仍残留大量的原油, 这些油或者被毛细管束缚住不能流动, 或者由于驱替相与被驱替相之间的不利流度比, 使得注入流波及体积小, 而无法驱动原油. 在注入液中加入某些化学添加剂, 则可大大改善注入液的驱洗油能力. 常用的化学添加剂大都为聚合物、表面活性剂和碱. 聚合物被用来优化驱替相的流速, 以调整与被驱相之间的流度比, 均匀驱动前缘, 减弱高渗层指进, 提高驱替相的波及效率, 同时增加压力梯度等. 表面活性剂和碱主要用于降低各相间的界面张力, 从而将被束缚的油启动^[1-6].

本文讨论在强化 (化学) 采油渗流力学数值模拟中提出的行之有效的迎风分数步差分方法的收敛性分析, 使得我们的软件系统建立在坚实的数学和力学基础上.

问题的数学模型是一类非线性耦合系统的初边值问题^[1-6]:

$$d(c) \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = q(X, t), \quad X = (x_1, x_2, x_3)^T \in \Omega, \quad t \in J = (0, T], \quad (1.1a)$$

$$\mathbf{u} = -a(c) \nabla p, \quad X \in \Omega, \quad t \in J, \quad (1.1b)$$

英文引用格式: Yuan Y R, Cheng A J, Yang D P, et al. Convergence analysis of an implicit upwind difference fractional steps method of three-dimensional enhanced oil recovery percolation coupled system (in Chinese). Sci Sin Math, 2014, 44: 1035–1058, doi: 10.1360/N012014-00009

$$\Phi(X) \frac{\partial c}{\partial t} + b(c) \frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla c - \nabla \cdot (D \nabla c) = g(X, t, c), \quad X \in \Omega, \quad t \in J, \quad (1.2)$$

$$\Phi(X) \frac{\partial}{\partial t}(cs_\alpha) + \nabla \cdot (s_\alpha \mathbf{u} - \phi c K_\alpha \nabla s_\alpha) = Q_\alpha(X, t, c, s_\alpha), \quad X \in \Omega, \quad t \in J, \quad \alpha = 1, 2, \dots, n_c, \quad (1.3)$$

其中 Ω 是有界区域, $a(c) = a(X, c) = k(X)\mu(c)^{-1}$, $d(c) = d(X, c)$, $\phi(X)$ 是岩石的孔隙度, $k(X)$ 是地层的渗透率, $\mu(c)$ 是流体的黏度, $D = D(X)$, $K_\alpha = K_\alpha(X)$ ($\alpha = 1, 2, \dots, n_c$) 均为相应的扩散系数, $\mathbf{u}$ 是 Darcy 速度, $p = p(X, t)$ 是压力函数, $c = c(X, t)$ 是水相饱和度函数, $s_\alpha = s_\alpha(X, t)$ 是组分浓度函数, 组分是指各种化学剂 (聚合物、表面活性剂、碱和各种离子等), n_c 是组分数.

二类边界条件.

(I) 定压边界条件:

$$p = e(X, t), \quad X \in \partial\Omega, \quad t \in J, \quad (1.4a)$$

$$c = h(X, t), \quad X \in \partial\Omega, \quad t \in J, \quad (1.4b)$$

$$s_\alpha = h_\alpha(X, t), \quad X \in \partial\Omega, \quad t \in J, \quad \alpha = 1, 2, \dots, n_c, \quad (1.4c)$$

其中 $\partial\Omega$ 为区域 Ω 的外边界面.

(II) 不渗透边界条件:

$$\mathbf{u} \cdot \gamma = 0, \quad X \in \partial\Omega, \quad t \in J, \quad (1.5a)$$

$$D \nabla c \cdot \gamma = 0, \quad X \in \partial\Omega, \quad t \in J, \quad (1.5b)$$

$$K_\alpha \nabla s_\alpha \cdot \gamma = 0, \quad X \in \partial\Omega, \quad t \in J, \quad \alpha = 1, 2, \dots, n_c, \quad (1.5c)$$

其中 γ 为边界面的单位外法向量.

初始条件:

$$p(X, 0) = p_0(X), \quad X \in \Omega, \quad (1.6a)$$

$$c(X, 0) = c_0(X), \quad X \in \Omega, \quad (1.6b)$$

$$s_\alpha(X, 0) = s_{\alpha,0}(X), \quad X \in \Omega, \quad \alpha = 1, 2, \dots, n_c. \quad (1.6c)$$

为了便于计算, 我们将方程 (1.3) 写为下述形式:

$$\begin{aligned} \phi c \frac{\partial}{\partial t}(s_\alpha) + \mathbf{u} \cdot \nabla s_\alpha - \nabla \cdot (\phi c K_\alpha \nabla s_\alpha) &= Q_\alpha - s_\alpha \left(q - d(c) \frac{\partial p}{\partial t} + \phi \frac{\partial c}{\partial t} \right), \\ X \in \Omega, \quad t \in J, \quad \alpha &= 1, 2, \dots, n_c. \end{aligned} \quad (1.7)$$

对平面不可压缩两相渗流驱动问题, 在问题的周期性假定下, 文献 [7-10] 等提出了特征差分方法和特征有限元法, 并给出了误差估计. 他们将特征线方法和标准的有限差分方法或有限元方法相结合, 真实地反映出对流 - 扩散方程的一阶双曲特性, 减少截断误差. 克服数值振荡和弥散, 大大提高了计算的稳定性和精确度. 对可压缩渗流驱动问题, 文献 [11-14] 同样在周期性假定下提出了二维可压缩渗流驱动问题的“微小压缩”数学模型、数值方法和分析, 开创了现代数值模型新领域. 袁益让^[15-17] 去掉周期性的假定, 给出了新的修正特征差分格式和有限元格式, 并得到了最佳的 l^2 模误差估计. 由于特征线法需要进行插值计算, 并且特征线在求解区域边界附近可能穿出边界, 需要作特殊处理. 特征

线与网格边界交点及其相应的函数值需要计算, 这样在算法设计时, 对靠近边界的网格点需要判断其特征线是否越过边界, 从而确定是否需要改变时间步长, 因此, 实际计算还是比较复杂的^[16, 17].

对抛物型问题, 文献 [18–20] 提出了迎风差分格式, 来克服数值解的振荡, 同时避免特征差分方法在对靠近边界网点的计算复杂性. 虽然, Douglas 和 Peaceman^[21] 曾将此方法用于不可压缩油水二相渗流驱动问题, 并取得了成功, 但在理论分析时出现实质性困难, 他们用 Fourier 分析方法仅能对常系数的情形证明稳定性和收敛性的结果, 此方法不能推广到变系数的情形^[22–24]. 本文从生产实际出发, 对三维可压缩二相渗流驱动强化采油渗流驱动耦合问题, 为克服计算复杂性, 提出一类隐式迎风分数步差分格式, 该格式既可克服数值振荡和弥散, 同时将三维问题化为连续解三个一维问题, 大大减少计算工作量, 使工程实际计算成为可能, 且将空间的计算精度提高到二阶. 本文应用变分形式、能量方法、差分算子乘积交替性理论、高阶差分算子的分解、微分方程先验估计和特殊的技巧, 得到了 l^2 模误差估计, 成功地解决了 Douglas 和 Ewing 所提的著名问题. 该方法已成功地应用到强化采油的数值模拟中^[4–6], 从而在能源数学领域起到一定程度的奠基作用.

通常问题是正定的, 即满足

$$0 < a_* \leq a(c) \leq a^*, \quad 0 < d_* \leq d(c) \leq d^*, \quad 0 < D_* \leq D(X, t) \leq D^*, \quad (1.8a)$$

$$0 < K_* \leq K_\alpha(X) \leq K^*, \quad \alpha = 1, 2, \dots, n_c,$$

$$\left| \frac{\partial a}{\partial c}(X, c) \right| \leq A^*, \quad (1.8b)$$

其中 a_* , a^* , d_* , d^* , D_* , D^* , K_* , K^* 和 A^* 均为正常数, $d(c)$, $b(c)$, $g(c)$ 和 $Q_\alpha(c, s_\alpha)$ 在解的 ε_0 邻域是 Lipschitz 连续的, 即存在正常数 M , 当 $|\varepsilon_i| \leq \varepsilon_0$ ($1 \leq i \leq 10$) 时有

$$|d(c(X, t) + \varepsilon_1) - d(c(X, t) + \varepsilon_2)| \leq M|\varepsilon_1 - \varepsilon_2|,$$

$$|b(c(X, t) + \varepsilon_3) - b(c(X, t) + \varepsilon_4)| \leq M|\varepsilon_3 - \varepsilon_4|,$$

$$|g(c(X, t) + \varepsilon_5) - g(c(X, t) + \varepsilon_6)| \leq M|\varepsilon_5 - \varepsilon_6|,$$

$$|Q_\alpha(c(X, t) + \varepsilon_7, s_\alpha(X, t) + \varepsilon_8) - Q_\alpha(c(X, t) + \varepsilon_9, s_\alpha(X, t) + \varepsilon_{10})| \leq M\{|\varepsilon_7 - \varepsilon_9| + |\varepsilon_8 - \varepsilon_{10}|\}.$$

假定问题 (1.1)–(1.6) 的精确解具有一定的光滑性, 即满足

$$p, c, s_\alpha \in L^\infty(W^{4,\infty}) \cap W^{1,\infty}(W^{1,\infty}), \quad \frac{\partial^2 p}{\partial t^2}, \frac{\partial^2 c}{\partial t^2}, \frac{\partial^2 s_\alpha}{\partial t^2} \in L^\infty(L^\infty), \quad \alpha = 1, 2, \dots, n_c.$$

本文记号 M 和 ε 分别表示普通正常数和普通小正数, 在不同处可具有不同的含义.

2 隐式迎风分数步差分格式

为了用差分方法求解, 我们用网格区域 Ω_h 代替 Ω (见图 1). 在空间 (x_1, x_2, x_3) 上 x_1 方向步长为 h_1 , x_2 方向步长为 h_2 , x_3 方向步长为 h_3 , $x_{1i} = ih_1$, $x_{2j} = jh_2$, $x_{3k} = kh_3$,

$$\Omega_h = \left\{ (x_{1i}, x_{2j}, x_{3k}) \left| \begin{array}{l} i_1(j, k) < i < i_2(j, k) \\ j_1(i, k) < j < j_2(i, k) \\ k_1(i, j) < k < k_2(i, j) \end{array} \right. \right\}.$$

 图 1 网域 Ω_h 的示意图

用 $\partial\Omega_h$ 表示 Ω_h 的边界. 为了记号简便, 以后记

$$X = (x_1, x_2, x_3)^T, \quad X_{ijk} = (ih_1, jh_2, kh_3)^T, \quad t^n = n\Delta t, \quad W(X_{ijk}, t^n) = W_{ijk}^n,$$

$$A_{i+1/2,jk}^n = \frac{1}{2}[a(X_{ijk}, C_{ijk}^n) + a(X_{i+1,jk}, C_{i+1,jk}^n)],$$

$$a_{i+1/2,jk}^n = \frac{1}{2}[a(X_{ijk}, c_{ijk}^n) + a(X_{i+1,jk}, c_{i+1,jk}^n)].$$

记号 $A_{i,j+1/2,k}^n$, $a_{i,j+1/2,k}^n$, $A_{ij,k+1/2}^n$ 和 $a_{ij,k+1/2}^n$ 的定义是类似的. 设

$$\delta_{\bar{x}_1}(A^n \delta_{x_1} P^{n+1})_{ijk} = h_1^{-2}[A_{i+1/2,jk}^n(P_{i+1,jk}^{n+1} - P_{ijk}^{n+1}) - A_{i-1/2,jk}^n(P_{ijk}^{n+1} - P_{i-1,jk}^{n+1})], \quad (2.1a)$$

$$\delta_{\bar{x}_2}(A^n \delta_{x_2} P^{n+1})_{ijk} = h_2^{-2}[A_{i,j+1/2,k}^n(P_{i,j+1,k}^{n+1} - P_{ijk}^{n+1}) - A_{i,j-1/2,k}^n(P_{ijk}^{n+1} - P_{i,j-1,k}^{n+1})], \quad (2.1b)$$

$$\delta_{\bar{x}_3}(A^n \delta_{x_3} P^{n+1})_{ijk} = h_3^{-2}[A_{ij,k+1/2}^n(P_{ij,k+1}^{n+1} - P_{ijk}^{n+1}) - A_{ij,k-1/2}^n(P_{ijk}^{n+1} - P_{ij,k-1}^{n+1})], \quad (2.1c)$$

$$\nabla_h(A^n \nabla_h P^{n+1})_{ijk} = \delta_{\bar{x}_1}(A^n \delta_{x_1} P^{n+1})_{ijk} + \delta_{\bar{x}_2}(A^n \delta_{x_2} P^{n+1})_{ijk} + \delta_{\bar{x}_3}(A^n \delta_{x_3} P^{n+1})_{ijk}. \quad (2.2)$$

流动方程 (1.1) 的分数步差分格式为

$$\begin{aligned} d(C_{ijk}^n) \frac{P_{ijk}^{n+1/3} - P_{ijk}^n}{\Delta t} &= \delta_{\bar{x}_1}(A^n \delta_{x_1} P^{n+1/3})_{ijk} + \delta_{\bar{x}_2}(A^n \delta_{x_2} P^n)_{ijk} \\ &\quad + \delta_{\bar{x}_3}(A^n \delta_{x_3} P^n)_{ijk} + q(X_{ijk}, t^{n+1}), \quad i_1(j, k) < i < i_2(j, k), \end{aligned} \quad (2.3a)$$

$$P_{ijk}^{n+1/3} = e_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (2.3b)$$

$$d(C_{ijk}^n) \frac{P_{ijk}^{n+2/3} - P_{ijk}^{n+1/3}}{\Delta t} = \delta_{\bar{x}_2}(A^n \delta_{x_2} (P^{n+2/3} - P^n))_{ijk}, \quad j_1(i, k) < j < j_2(i, k), \quad (2.3c)$$

$$P_{ijk}^{n+2/3} = e_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (2.3d)$$

$$d(C_{ijk}^n) \frac{P_{ijk}^{n+1} - P_{ijk}^{n+2/3}}{\Delta t} = \delta_{\bar{x}_3}(A^n \delta_{x_3} (P^{n+1} - P^n))_{ijk}, \quad k_1(i, j) < k < k_2(i, j), \quad (2.3e)$$

$$P_{ijk}^{n+1} = e_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h. \quad (2.3f)$$

近似 Darcy 速度 $\mathbf{U}^n = (U_1^n, U_2^n, U_3^n)^T$ 按下述公式计算:

$$U_{1,ijk}^{n+1} = -\frac{1}{2} \left[A_{i+1/2,jk}^n \frac{P_{i+1,jk}^{n+1} - P_{ijk}^{n+1}}{h_1} + A_{i-1/2,jk}^n \frac{P_{ijk}^{n+1} - P_{i-1,jk}^{n+1}}{h_1} \right], \quad (2.4)$$

对应于另外两个方向的速度 $U_{2,ijk}^{n+1}$ 和 $U_{3,ijk}^{n+1}$ 可类似计算.

下面考虑饱和度方程 (1.2) 的隐式迎风分数步计算格式,

$$\begin{aligned} \phi_{ijk} \frac{C_{ijk}^{n+1/3} - C_{ijk}^n}{\Delta t} = & \left(1 + \frac{h_1}{2} |U_1^{n+1}| D^{-1}\right)_{ijk}^{-1} \delta_{\bar{x}_1} (D \delta_{x_1} C^{n+1/3})_{ijk} + \left(1 + \frac{h_2}{2} |U_2^{n+1}| D^{-1}\right)_{ijk}^{-1} \\ & \times \delta_{\bar{x}_2} (D \delta_{x_2} C^n)_{ijk} + \left(1 + \frac{h_3}{2} |U_3^{n+1}| D^{-1}\right)_{ijk}^{-1} \delta_{\bar{x}_3} (D \delta_{x_3} C^n)_{ijk} \\ & - b(C_{ijk}^n) \frac{P_{ijk}^{n+1} - P_{ijk}^n}{\Delta t} + f(X_{ijk}, t^n, C_{ijk}^n), \quad i_1(j, k) < i < i_2(j, k), \end{aligned} \quad (2.5a)$$

$$C_{ijk}^{n+1/3} = h_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (2.5b)$$

$$\phi_{ijk} \frac{C_{ijk}^{n+2/3} - C_{ijk}^{n+1/3}}{\Delta t} = \left(1 + \frac{h_2}{2} |U_2^{n+1}| D^{-1}\right)_{ijk}^{-1} \delta_{\bar{x}_2} (D \delta_{x_2} C^n)_{ijk}, \quad j_1(i, k) < j < j_2(i, k), \quad (2.5c)$$

$$C_{ijk}^{n+2/3} = h_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (2.5d)$$

$$\begin{aligned} \phi_{ijk} \frac{C_{ijk}^{n+1} - C_{ijk}^{n+2/3}}{\Delta t} = & \left(1 + \frac{h_3}{2} |U_3^{n+1}| D^{-1}\right)_{ijk}^{-1} \delta_{\bar{x}_3} (D \delta_{x_3} C^n)_{ijk} \\ & - \sum_{\beta=1}^3 \delta_{U_{\beta}^{n+1}, x_{\beta}} C_{ijk}^{n+1}, \quad k_1(i, j) < k < k_2(i, j), \end{aligned} \quad (2.5e)$$

$$C_{ijk}^{n+1} = h_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (2.5f)$$

其中

$$\begin{aligned} \delta_{U_1^{n+1}, x_1} C_{ijk}^{n+1} &= U_{1,ijk}^{n+1} \{H(U_{1,ijk}^{n+1}) D_{ijk}^{-1} D_{i-1/2,jk} \delta_{\bar{x}_1} + (1 - H(U_{1,ijk}^{n+1})) D_{ijk}^{-1} D_{i+1/2,jk} \delta_{x_1}\} C_{ijk}^{n+1}, \\ \delta_{U_2^{n+1}, x_2} C_{ijk}^{n+1} &= U_{2,ijk}^{n+1} \{H(U_{2,ijk}^{n+1}) D_{ijk}^{-1} D_{i,j-1/2,k} \delta_{\bar{x}_2} + (1 - H(U_{2,ijk}^{n+1})) D_{ijk}^{-1} D_{i,j+1/2,k} \delta_{x_2}\} C_{ijk}^{n+1}, \\ \delta_{U_3^{n+1}, x_3} C_{ijk}^{n+1} &= U_{3,ijk}^{n+1} \{H(U_{3,ijk}^{n+1}) D_{ijk}^{-1} D_{ij,k-1/2} \delta_{\bar{x}_3} + (1 - H(U_{3,ijk}^{n+1})) D_{ijk}^{-1} D_{ij,k+1/2} \delta_{x_3}\} C_{ijk}^{n+1}, \end{aligned}$$

以及

$$H(z) = \begin{cases} 1, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

对组分浓度方程 (1.7) 亦采用隐式迎风分数步差分并行计算,

$$\begin{aligned} \phi_{ijk} C_{ijk}^{m+1} \frac{S_{\alpha,ijk}^{n+1/3} - S_{\alpha,ijk}^n}{\Delta t} &= \delta_{\bar{x}_1} (C_{ijk}^{m+1} \phi K_{\alpha} \delta_{x_1} S_{\alpha}^{n+1/3})_{ijk} + \delta_{\bar{x}_2} (C_{ijk}^{m+1} \phi K_{\alpha} \delta_{x_2} S_{\alpha}^n)_{ijk} + \delta_{\bar{x}_3} (C_{ijk}^{m+1} \phi K_{\alpha} \delta_{x_3} S_{\alpha}^n)_{ijk} \\ &+ Q_{\alpha}(C_{ijk}^{m+1}, S_{\alpha,ijk}^n) - S_{\alpha,ijk}^n \left(q(C_{ijk}^{m+1}) - d(C_{ijk}^{m+1}) \frac{P_{ijk}^{n+1} - P_{ijk}^n}{\Delta t} \right. \\ &\left. + \phi_{ijk} \frac{C_{ijk}^{m+1} - C_{ijk}^n}{\Delta t} \right), \quad i_1(j, k) < i < i_2(j, k), \quad \alpha = 1, 2, \dots, n_c, \end{aligned} \quad (2.6a)$$

$$S_{\alpha,ijk}^{n+1/3} = h_{\alpha,ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad \alpha = 1, 2, \dots, n_c, \quad (2.6b)$$

$$\phi_{ijk} C_{ijk}^{m+1} \frac{S_{\alpha,ijk}^{n+2/3} - S_{\alpha,ijk}^{n+1/3}}{\Delta t} = \delta_{\bar{x}_2} (C_{ijk}^{m+1} \phi K_{\alpha} \delta_{x_2} (S_{\alpha}^{n+1/3} - S_{\alpha}^n))_{ijk},$$

$$j_1(i, k) < j < j_2(i, k), \quad \alpha = 1, 2, \dots, n_c, \quad (2.6c)$$

$$S_{\alpha,ijk}^{n+2/3} = h_{\alpha,ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad \alpha = 1, 2, \dots, n_c, \quad (2.6d)$$

$$\phi_{ijk} C_{ijk}^{n+1} \frac{S_{\alpha,ijk}^{n+1} - S_{\alpha,ijk}^{n+2/3}}{\Delta t} = \delta_{\bar{x}_3} (C^{n+1} \phi K_{\alpha} \delta_{x_3} (S_{\alpha}^{n+1} - S_{\alpha}^n))_{ijk} - \sum_{\beta=1}^3 \bar{\delta}_{U_{\beta}^{n+1}, x_{\beta}} S_{\alpha,ijk}^{n+1},$$

$$k_1(i, j) < k < k_2(i, j), \quad \alpha = 1, 2, \dots, n_c, \quad (2.6e)$$

$$S_{\alpha,ijk}^{n+1} = h_{\alpha,ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad \alpha = 1, 2, \dots, n_c, \quad (2.6f)$$

其中

$$\bar{\delta}_{U_{\beta}^{n+1}, x_{\beta}} = U_{\beta,ijk}^{n+1} \{H(U_{\beta,ijk}^{n+1}) \delta_{\bar{x}_{\beta}} + (1 - H(U_{\beta,ijk}^{n+1})) \delta_{x_{\beta}}\}, \quad \beta = 1, 2, 3.$$

初始条件:

$$P_{ijk}^0 = p_0(X_{ijk}), \quad C_{ijk}^0 = c_0(X_{ijk}), \quad S_{\alpha,ijk}^0 = s_{\alpha,0}(X_{ijk}), \quad X_{ijk} \in \Omega_h, \quad \alpha = 1, 2, \dots, n_c. \quad (2.7)$$

隐式迎风分数步差分格式的计算程序是, 当 $\{P_{ijk}^n, C_{ijk}^n, S_{\alpha,ijk}^n, \alpha = 1, 2, \dots, n_c\}$ 已知, 首先由 (2.3a) 和 (2.3b) 沿 x_1 方向用追赶法求出过渡层的解 $\{P_{ijk}^{n+1/3}\}$, 再由 (2.3c) 和 (2.3d) 沿 x_2 方向用追赶法求出 $\{P_{ijk}^{n+2/3}\}$, 最后由 (2.3e) 和 (2.3f) 沿 x_3 方向用追赶法求出解 $\{P_{ijk}^{n+1}\}$, 应用 (2.4) 计算出 $\{U_{ijk}^{n+1}\}$. 其次由 (2.5a) 和 (2.5b) 沿 x_1 方向用追赶法求出过渡层的解 $\{C_{ijk}^{n+1/3}\}$, 再由 (2.5c) 和 (2.5d) 沿 x_2 方向用追赶法求出 $\{C_{ijk}^{n+2/3}\}$, 最后再由 (2.5e) 和 (2.5f) 沿 x_3 方向用追赶法求出解 $\{C_{ijk}^{n+1}\}$. 在此基础并行地由 (2.6a) 和 (2.6b) 沿 x_1 方向用追赶法求出过渡层的解 $\{S_{\alpha,ijk}^{n+1/3}\}$, 再由 (2.6c) 和 (2.6d) 沿 x_2 方向用追赶法求出 $\{S_{\alpha,ijk}^{n+2/3}\}$, 最后由 (2.6e) 和 (2.6f) 沿 x_3 方向用追赶法求出解 $\{S_{\alpha,ijk}^{n+1}\}$. 对 $\alpha = 1, 2, \dots, n_c$ 可并行地同时求解. 由问题的正定性, 格式 (2.3)、(2.5) 和 (2.6) 的解存在且唯一.

3 收敛性分析

为了分析方便, 设区域 $\Omega = \{[0, 1]\}^3$, $h = 1/N$, $X_{ijk} = (x_1, x_2, x_3)^T = (ih, jh, kh)^T$, $t^n = n\Delta t$, $W(X_{ijk}, t^n) = W_{ijk}^n$. 设 $\pi = p - P$, $\xi = c - C$, $\zeta_{\alpha} = s_{\alpha} - S_{\alpha}$, 其中 p , c 和 s_{α} 为问题 (1.1)–(1.6) 的精确解, P , C 和 S_{α} 为格式 (2.3)–(2.6) 的差分解. 为了进行误差分析, 引入对应于 $l^2(\Omega)$ 和 $H^1(\Omega)$ 的内积和范数^[25, 26],

$$\begin{aligned} \langle v^n, w^n \rangle &= \sum_{i,j,k=1}^N v_{ijk}^n w_{ijk}^n h^3, \quad \|v^n\|_0^2 = \langle v^n, v^n \rangle, \quad [v^n, w^n]_1 = \sum_{i=0}^{N-1} \sum_{j,k=1}^N v_{ijk}^n w_{ijk}^n h^3, \\ [v^n, w^n]_2 &= \sum_{j=0}^{N-1} \sum_{i,k=1}^N v_{ijk}^n w_{ijk}^n h^3, \quad [v^n, w^n]_3 = \sum_{k=0}^{N-1} \sum_{i,j=1}^N v_{ijk}^n w_{ijk}^n h^3, \\ \|\delta_{x_1} v^n\|^2 &= [\delta_{x_1} v^n, \delta_{x_1} v^n]_1, \quad \|\delta_{x_2} v^n\|^2 = [\delta_{x_2} v^n, \delta_{x_2} v^n]_2, \quad \|\delta_{x_3} v^n\|^2 = [\delta_{x_3} v^n, \delta_{x_3} v^n]_3. \end{aligned}$$

定理 1 假定问题 (1.1)–(1.6) 的精确解满足光滑性条件: $p, c \in W^{1,\infty}(W^{1,\infty}) \cap L^{\infty}(W^{4,\infty})$, $s_{\alpha} \in W^{1,\infty}(W^{1,\infty}) \cap L^{\infty}(W^{3,\infty})$, $\frac{\partial p}{\partial t}, \frac{\partial c}{\partial t} \in L^{\infty}(W^{4,\infty})$, $\frac{\partial s_{\alpha}}{\partial t} \in L^{\infty}(W^{3,\infty})$, $\frac{\partial^2 p}{\partial t^2}, \frac{\partial^2 c}{\partial t^2}, \frac{\partial^2 s_{\alpha}}{\partial t^2} \in L^{\infty}(L^{\infty})$, $\alpha = 1, 2, \dots, n_c$. 采用修正迎风分数步差分格式 (2.3)–(2.6) 逐层计算, 若剖分参数满足如下限制性条件:

$$\Delta t \leq Mh^2, \quad (3.1)$$

则下述误差估计式成立,

$$\begin{aligned} & \|p - P\|_{\bar{L}^\infty((0,T];h^1)} + \|c - C\|_{\bar{L}^\infty((0,T];h^1)} + \|d_t(p - P)\|_{\bar{L}^2((0,T];l^2)} + \|d_t(c - C)\|_{\bar{L}^2((0,T];l^2)} \\ & \leq M_1^* \{\Delta t + h^2\}, \end{aligned} \quad (3.2a)$$

$$\|s_\alpha - S_\alpha\|_{\bar{L}^\infty((0,T];h^1)} + \|d_t(s_\alpha - S_\alpha)\|_{\bar{L}^2((0,T];l^2)} \leq M_2^* \{\Delta t + h\}, \quad \alpha = 1, 2, \dots, n_c, \quad (3.2b)$$

其中 $\|g\|_{\bar{L}^\infty(J;\mathcal{M})} = \sup_{n\Delta t \leq T} \|g^n\|_{\mathcal{M}}$, 常数

$$\begin{aligned} M_1^* &= M_1^* \left(\|p\|_{W^{1,\infty}(W^{4,\infty})}, \|p\|_{L^\infty(W^{4,\infty})}, \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(W^{4,\infty})}, \right. \\ & \quad \left. \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^\infty(L^\infty)}, \|c\|_{W^{1,\infty}(W^{4,\infty})}, \left\| \frac{\partial c}{\partial t} \right\|_{L^\infty(W^{4,\infty})}, \left\| \frac{\partial^2 c}{\partial t^2} \right\|_{L^\infty(L^\infty)} \right), \\ M_2^* &= M_2^* \left(\|s_\alpha\|_{W^{1,\infty}(W^{4,\infty})}, \left\| \frac{\partial s_\alpha}{\partial t} \right\|_{L^\infty(W^{4,\infty})}, \left\| \frac{\partial^2 s_\alpha}{\partial t^2} \right\|_{L^\infty(L^\infty)} \right). \end{aligned}$$

证明 首先研究流动方程, 由 (2.3a)、(2.3c) 和 (2.3e) 消去 $P^{n+1/3}$ 和 $P^{n+2/3}$ 可得到下述等价的差分方程:

$$\begin{aligned} & d(C_{ijk}^n) \frac{P_{ijk}^{n+1} - P_{ijk}^n}{\Delta t} - \nabla_h(A^n \nabla_h P^{n+1})_{ijk} \\ & = q(X_{ijk}, t^{n+1}) - (\Delta t)^2 \{ \delta_{\bar{x}_1}(A^n \delta_{x_1}(d^{-1}(C^n) \delta_{x_2}(A^n \delta_{x_2} d_t P^n)))_{ijk} \\ & \quad + \delta_{\bar{x}_1}(A^n \delta_{x_1}(d^{-1}(C^n) \delta_{x_3}(A^n \delta_{x_3} d_t P^n)))_{ijk} + \delta_{\bar{x}_2}(A^n \delta_{x_2}(d^{-1}(C^n) \delta_{x_3}(A^n \delta_{x_3} d_t P^n)))_{ijk} \} \\ & \quad + (\Delta t)^3 \delta_{\bar{x}_1}(A^n \delta_{x_1}(d^{-1}(C^n) \delta_{x_2}(A^n \delta_{x_2}(d^{-1}(C^n) \delta_{x_3}(A^n \delta_{x_3} d_t P^n))))_{ijk}, \quad 1 \leq i, j, k \leq N-1, \end{aligned} \quad (3.3a)$$

$$P_{ijk}^{n+1} = e_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (3.3b)$$

其中

$$d_t P_{ijk}^n = \frac{P_{ijk}^{n+1} - P_{ijk}^n}{\Delta t}.$$

由流动方程 (1.1) ($t = t^{n+1}$) 和差分方程 (3.3) 相减可得压力函数的误差方程

$$\begin{aligned} & d(C_{ijk}^n) \frac{\pi_{ijk}^{n+1} - \pi_{ijk}^n}{\Delta t} - \nabla_h(A^n \nabla_h \pi^{n+1})_{ijk} \\ & = -[d(c_{ijk}^{n+1}) - d(C_{ijk}^n)] \frac{p_{ijk}^{n+1} - p_{ijk}^n}{\Delta t} + \nabla_h([a(c^{n+1}) - a(C^n)] \nabla_h p^{n+1})_{ijk} \\ & \quad - (\Delta t)^2 \{ [\delta_{\bar{x}_1}(a^{n+1} \delta_{x_1}(d^{-1}(c^{n+1}) \delta_{x_2}(a^{n+1} \delta_{x_2} d_t p^n)))_{ijk} - \delta_{\bar{x}_1}(A^n \delta_{x_1}(d^{-1}(C^n) \delta_{x_2}(A^n \delta_{x_2} d_t P^n)))_{ijk}] \\ & \quad + [\delta_{\bar{x}_1}(a^{n+1} \delta_{x_1}(d^{-1}(c^{n+1}) \delta_{x_3}(a^{n+1} \delta_{x_3} d_t p^n)))_{ijk} - \delta_{\bar{x}_1}(A^n \delta_{x_1}(d^{-1}(C^n) \delta_{x_3}(A^n \delta_{x_3} d_t P^n)))_{ijk}] \\ & \quad + [\delta_{\bar{x}_2}(a^{n+1} \delta_{x_2}(d^{-1}(c^{n+1}) \delta_{x_3}(a^{n+1} \delta_{x_3} d_t p^n)))_{ijk} - \delta_{\bar{x}_2}(A^n \delta_{x_2}(d^{-1}(C^n) \delta_{x_3}(A^n \delta_{x_3} d_t P^n)))_{ijk}] \} \\ & \quad + (\Delta t)^3 \{ \delta_{\bar{x}_1}(a^{n+1} \delta_{x_1}(d^{-1}(c^{n+1}) \delta_{x_2}(a^{n+1} \delta_{x_2}(d^{-1}(c^{n+1}) \delta_{x_3}(a^{n+1} \delta_{x_3} d_t p^n))))_{ijk} - \delta_{\bar{x}_1}(A^n \delta_{x_1} \\ & \quad \times (d^{-1}(C^n) \delta_{x_2}(A^n \delta_{x_2}(d^{-1}(C^n) \delta_{x_3}(A^n \delta_{x_3} d_t P^n))))_{ijk} + \sigma_{ijk}^{n+1}, \quad 1 \leq i, j, k \leq N-1, \end{aligned} \quad (3.4a)$$

$$\pi_{ijk}^{n+1} = 0, \quad X_{ijk} \in \partial\Omega_h, \quad (3.4b)$$

其中

$$|\sigma_{ijk}^{n+1}| \leq M \left(\left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^\infty(L^\infty)}, \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(W^{4,\infty})}, \|p\|_{L^\infty(W^{4,\infty})}, \|c\|_{L^\infty(W^{3,\infty})} \right) \{\Delta t + h^2\}.$$

引入归纳法假定

$$\sup_{1 \leq n \leq L} \max\{\|\pi^n\|_{1,\infty}, \|\xi^n\|_{1,\infty}\} \rightarrow 0, \quad (h, \Delta t) \rightarrow 0, \quad (3.5)$$

其中 $\|\pi^n\|_{1,\infty}^2 = |\pi^n|_{0,\infty}^2 + |\nabla_h \pi^n|_{0,\infty}^2$.

对误差方程 (3.4) 利用变分形式, 乘以 $\delta_t \pi_{ijk}^n = d_t \pi_{ijk}^n \Delta t = \pi_{ijk}^{n+1} - \pi_{ijk}^n$ 作内积, 并应用分部求和公式可得

$$\begin{aligned} & \langle d(C^n) d_t \pi^n, d_t \pi^n \rangle \Delta t + \frac{1}{2} \{ \langle A^n \nabla_h \pi^{n+1}, \nabla_h \pi^{n+1} \rangle - \langle A^n \nabla_h \pi^n, \nabla_h \pi^n \rangle \} \\ & \leq - \langle [d(c^{n+1}) - d(C^n)] d_t p^n, d_t \pi^n \rangle \Delta t + \langle \nabla_h ([a(c^{n+1}) - a(C^n)] \nabla_h p^{n+1}), d_t \pi^n \rangle \Delta t \\ & \quad - (\Delta t)^3 \{ \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} d_t p^n))) - \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} d_t P^n))), d_t \pi^n \rangle \\ & \quad + \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_3} (a^{n+1} \delta_{x_3} d_t p^n))) - \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_3} (A^n \delta_{x_3} d_t P^n))), d_t \pi^n \rangle \\ & \quad + \langle \delta_{\bar{x}_2} (a^{n+1} \delta_{x_2} (d^{-1}(c^{n+1}) \delta_{x_3} (a^{n+1} \delta_{x_3} d_t p^n))) - \delta_{\bar{x}_2} (A^n \delta_{x_2} (d^{-1}(C^n) \delta_{x_3} (A^n \delta_{x_3} d_t P^n))), d_t \pi^n \rangle \} \\ & \quad + (\Delta t)^4 \{ \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} (d^{-1}(c^{n+1}) \delta_{x_3} (a^{n+1} \delta_{x_3} d_t p^n)))) \\ & \quad - \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} (d^{-1}(C^n) \delta_{x_3} (A^n \delta_{x_3} d_t P^n))))), d_t \pi^n \rangle \} + \langle \sigma^{n+1}, d_t \pi^n \rangle \Delta t. \end{aligned} \quad (3.6)$$

依次估计 (3.6) 右端诸项. (3.6) 右端第一项和第二项分别为

$$- \langle [d(c^{n+1}) - d(C^n)] d_t p^n, d_t \pi^n \rangle \Delta t \leq M \{ \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \pi^n\|^2 \Delta t, \quad (3.7a)$$

$$\langle \nabla_h ([a(c^{n+1}) - a(C^n)] \nabla_h p^{n+1}), d_t \pi^n \rangle \Delta t \leq M \{ \|\nabla_h \xi^n\|^2 + \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \pi^n\|^2 \Delta t. \quad (3.7b)$$

下面讨论 (3.6) 右端第三项, 首先分析其第一部分:

$$\begin{aligned} & - (\Delta t)^3 \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} d_t p^n))) - \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} d_t P^n))), d_t \pi^n \rangle \\ & = - (\Delta t)^3 \{ \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} d_t \pi^n))), d_t \pi^n \rangle + \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(c^n) \delta_{x_2} ([a^{n+1} \\ & \quad - A^n] \delta_{x_2} d_t p^n))), d_t \pi^n \rangle + \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} ([d^{-1}(c^{n+1}) - d(C^n)] \delta_{x_2} (a^{n+1} \delta_{x_2} d_t p^n))), d_t \pi^n \rangle \\ & \quad + \langle \delta_{\bar{x}_1} ([a^{n+1} - A^n] \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} d_t P^n))), d_t \pi^n \rangle \}. \end{aligned} \quad (3.7c)$$

现重点讨论 (3.7c) 右端式的第一项, 尽管 $-\delta_{\bar{x}_1} (A^n \delta_{x_1}), -\delta_{\bar{x}_2} (A^n \delta_{x_2}), \dots$ 是自共轭、正定和有界算子, 且空间区域为单位正立方体, 但它们的乘积一般是不可交替的, 利用差分算子乘积交换性 $\delta_{x_1} \delta_{x_2} = \delta_{x_2} \delta_{x_1}$, $\delta_{\bar{x}_1} \delta_{x_2} = \delta_{x_2} \delta_{\bar{x}_1}$, $\delta_{x_1} \delta_{\bar{x}_2} = \delta_{\bar{x}_2} \delta_{x_1}$, $\delta_{\bar{x}_1} \delta_{\bar{x}_2} = \delta_{\bar{x}_2} \delta_{\bar{x}_1}$, 有

$$\begin{aligned} & - (\Delta t)^3 \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} d_t \pi^n))), d_t \pi^n \rangle \\ & = (\Delta t)^3 \langle A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} d_t \pi^n)), \delta_{\bar{x}_1} d_t \pi^n \rangle \\ & = (\Delta t)^3 \langle d^{-1}(C^n) \delta_{x_1} \delta_{x_2} (A^n \delta_{x_2} d_t \pi^n) + \delta_{x_1} d^{-1}(C^n) \cdot \delta_{x_2} (A^n \delta_{x_2} d_t \pi^n), A^n \delta_{\bar{x}_1} d_t \pi^n \rangle \\ & = (\Delta t)^3 \{ \langle \delta_{\bar{x}_2} \delta_{x_1} (A^n \delta_{x_2} d_t \pi^n), d^{-1}(C^n) A^n \delta_{x_1} d_t \pi^n \rangle + \langle \delta_{\bar{x}_2} (A^n \delta_{\bar{x}_2} d_t \pi^n), \delta_{x_1} d^{-1}(C^n) \cdot A^n \delta_{\bar{x}_1} d_t \pi^n \rangle \} \\ & = - (\Delta t)^3 \{ \langle \delta_{\bar{x}_1} (A^n \delta_{x_2} d_t \pi^n), \delta_{x_2} (d^{-1}(C^n) A^n \delta_{x_1} d_t \pi^n) \rangle + \langle A^n \delta_{x_2} d_t \pi^n, \delta_{x_2} (\delta_{x_1} d^{-1}(C^n) \cdot A^n \delta_{x_1} d_t \pi^n) \rangle \} \\ & = - (\Delta t)^3 \{ \langle A^n \delta_{x_1} \delta_{x_2} d_t \pi^n + \delta_{x_1} A^n \cdot \delta_{x_2} d_t \pi^n, d^{-1}(C^n) A^n \delta_{x_1} \delta_{x_2} d_t \pi^n + \delta_{x_2} (d^{-1}(C^n) A^n) \cdot \delta_{x_1} d_t \pi^n \rangle \\ & \quad + \langle A^n \delta_{x_2} d_t \pi^n, \delta_{x_2} \delta_{x_1} d^{-1}(C^n) \cdot A^n \delta_{x_1} d_t \pi^n + \delta_{x_1} d^{-1}(C^n) \delta_{x_2} A^n \cdot \delta_{x_1} d_t \pi^n \\ & \quad + \delta_{x_1} d^{-1}(C^n) A^n \delta_{x_1} \delta_{x_2} d_t \pi^n \rangle \} \end{aligned}$$

$$\begin{aligned}
&= -(\Delta t)^3 \sum_{i,j,k=1}^N \{A_{i,j+1/2,k}^n A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n) [\delta_{x_1} \delta_{x_2} d_t \pi_{ijk}^n]^2 \\
&\quad + [A_{i,j+1/2,k}^n \delta_{x_2} (A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n)) \cdot \delta_{x_1} d_t \pi_{ijk}^n + A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n) \delta_{x_1} A_{i,j+1/2,k}^n \cdot \delta_{x_2} d_t \pi_{ijk}^n \\
&\quad + A_{i,j+1/2,k}^n A_{i+1/2,jk}^n \delta_{x_1} d^{-1}(C_{ijk}^n) \delta_{x_2} d_t \pi_{ijk}^n] \delta_{x_1} \delta_{x_2} d_t \pi_{ijk}^n + [\delta_{x_1} A_{i,j+1/2,k}^n \cdot \delta_{x_2} (d^{-1}(C_{ijk}^n) A_{i+1/2,jk}^n) \\
&\quad + A_{i,j+1/2,k}^n \delta_{x_2} A_{i+1/2,jk}^n \delta_{x_1} d^{-1}(C_{ijk}^n) + A_{i,j+1/2,k}^n \cdot A_{i+1/2,jk}^n \delta_{x_1} \delta_{x_2} d^{-1}(C_{ijk}^n)] \\
&\quad \times \delta_{x_2} d_t \pi_{ijk}^n \delta_{x_1} d_t \pi_{ijk}^n \} h^3.
\end{aligned} \tag{3.8}$$

由归纳法假定 (3.5) 可以推出 $A_{i+1/2,jk}^n$, $A_{i,j+1/2,k}^n$, $d^{-1}(C_{ijk}^n)$, $\delta_{x_2}(A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n))$ 和 $\delta_{x_1} A_{i,j+1/2,k}^n$ 是有界的. 对上述表达式的前两项, 应用 A, d^{-1} 的正定性和高阶差分算子的分解, 可分离出高阶差商项 $\delta_{x_1} \delta_{x_2} d_t \pi^n$, 即利用 Cauchy 不等式消去与此有关的项可得

$$\begin{aligned}
&-(\Delta t)^3 \sum_{i,j,k=1}^N \{A_{i,j+1/2,k}^n A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n) [\delta_{x_1} \delta_{x_2} d_t \pi_{ijk}^n]^2 \\
&\quad + [A_{i,j+1/2,k}^n \delta_{x_2} (A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n)) \cdot \delta_{x_1} d_t \pi_{ijk}^n + A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n) \delta_{x_1} A_{i,j+1/2,k}^n \cdot \delta_{x_2} d_t \pi_{ijk}^n \\
&\quad + \cdots] \delta_{x_1} \delta_{x_2} d_t \pi_{ijk}^n \} h^3 \\
&\leq -(\Delta t)^3 \sum_{i,j,k=1}^N \{a_*^2 (d^*)^{-1} [\delta_{x_1} \delta_{x_2} d_t \pi_{ijk}^n]^2 + [A_{i,j+1/2,k}^n \delta_{x_2} (A_{i+1/2,jk}^n d^{-1}(C_{ijk}^n)) \cdot \delta_{x_1} d_t \pi_{ijk}^n \\
&\quad + \cdots] \delta_{x_1} \delta_{x_2} d_t \pi_{ijk}^n \} h^3 \\
&\leq M\{[\|\delta_{x_1} d_t \pi^n\|^2 + \|\delta_{x_2} d_t \pi^n\|^2]\} (\Delta t)^3 \\
&\leq M \Delta t \{ \|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \pi^n\|^2 \}.
\end{aligned} \tag{3.9a}$$

对 (3.8) 的第三项有

$$\begin{aligned}
&-(\Delta t)^3 \sum_{i,j,k=1}^N [\delta_{x_1} A_{i,j+1/2,k}^n \cdot \delta_{x_2} (d^{-1}(C_{ijk}^n) A_{i+1/2,jk}^n) \\
&\quad + A_{i,j+1/2,k}^n \delta_{x_2} A_{i+1/2,jk}^n \delta_{x_1} d^{-1}(C_{ijk}^n)] \delta_{x_1} d_t \pi_{ijk}^n \delta_{x_2} d_t \pi_{ijk}^n h^3 \\
&\leq M\{ \|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \pi^n\|^2 \} \Delta t,
\end{aligned} \tag{3.9b}$$

$$\begin{aligned}
&-(\Delta t)^3 \sum_{i,j,k=1}^N A_{i,j+1/2,k}^n A_{i+1/2,jk}^n \delta_{x_1} \delta_{x_2} d^{-1}(C_{ijk}^n) \delta_{x_1} d_t \pi_{ijk}^n \delta_{x_2} d_t \pi_{ijk}^n h^3 \\
&\leq M(\Delta t)^{1/2} \|d_t \pi^n\|^2 \Delta t \leq \varepsilon \|d_t \pi^n\|^2 \Delta t,
\end{aligned} \tag{3.9c}$$

于是得到

$$\begin{aligned}
&-(\Delta t)^3 \{ \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} d_t \pi^n))), d_t \pi^n \rangle \\
&\quad \leq M\{ \|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \pi^n\|^2 \} \Delta t + \varepsilon \|d_t \pi^n\|^2 \Delta t.
\end{aligned} \tag{3.10}$$

对 (3.7c) 的其余项可类似的讨论得到

$$-(\Delta t)^3 \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} d_t p^n))), d_t \pi^n \rangle$$

$$\leq M\{\|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \pi^n\|^2 + \|\xi^n\|^2 + (\Delta t)^2\} \Delta t + \varepsilon \|d_t \pi^n\|^2 \Delta t. \quad (3.11)$$

类似地, 对 (3.6) 中右端第三项的其余两项亦有相同的估计式 (3.11).

对于 (3.6) 中右端第四项, 有

$$\begin{aligned} & (\Delta t)^4 \{ \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} (d^{-1}(c^{n+1}) \delta_{x_3} (a^{n+1} \delta_{x_3} d_t p^n)))) \\ & \quad - \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} (d^{-1}(C^n) \delta_{x_3} (A^n \delta_{x_3} d_t P^n))))), d_t \pi^n \rangle \} \\ & = (\Delta t)^4 \{ \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} (d^{-1}(C^n) \delta_{x_3} (A^n \delta_{x_3} d_t \pi^n))))), d_t \pi^n \rangle \\ & \quad + \langle \delta_{\bar{x}_1} (a^{n+1} \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} (d^{-1}(c^{n+1}) \delta_{x_3} ([a^{n+1} - A^n] \delta_{x_3} d_t p^n))))), d_t \pi^n \rangle \\ & \quad + \cdots + \langle \delta_{\bar{x}_1} ([a^{n+1} - A^n] \delta_{x_1} (d^{-1}(c^{n+1}) \delta_{x_2} (a^{n+1} \delta_{x_2} (d^{-1}(c^{n+1}) \delta_{x_3} (a^{n+1} \delta_{x_3} d_t p^n))))), d_t \pi^n \rangle \}. \end{aligned}$$

对上式经类似的分析和计算可得

$$\begin{aligned} & (\Delta t)^4 \{ \langle \delta_{\bar{x}_1} (A^n \delta_{x_1} (d^{-1}(C^n) \delta_{x_2} (A^n \delta_{x_2} (d^{-1}(C^n) \delta_{x_3} (A^n \delta_{x_3} d_t \pi^n))))), d_t \pi^n \rangle + \cdots \} \\ & \leq -\frac{1}{2} a_*^3 (d^*)^{-2} (\Delta t)^4 \sum_{i,j,k=1}^N [\delta_{x_1} \delta_{x_2} \delta_{x_3} d_t \pi_{ijk}^n]^2 h^3 + M\{\|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \pi^n\|^2 \\ & \quad + \|\xi^n\|^2 + (\Delta t)^2\} \Delta t + \varepsilon \|d_t \pi^n\|^2 \Delta t. \end{aligned} \quad (3.12)$$

当 Δt 适当小时, ε 可适当小. 对误差方程 (3.6), 应用 (3.7)–(3.12), 经计算可得

$$\begin{aligned} & \|d_t \pi^n\|^2 \Delta t + \frac{1}{2} \{ \langle A^n \nabla_h \pi^{n+1}, \nabla_h \pi^{n+1} \rangle - \langle A^n \nabla_h \pi^n, \nabla_h \pi^n \rangle \} \\ & \leq M\{\|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \pi^n\|^2 + h^4 + (\Delta t)^2\} \Delta t. \end{aligned} \quad (3.13)$$

下面讨论饱和度方程的误差估计. 由 (2.5a)、(2.5c) 和 (2.5e) 消去 $C_{ijk}^{n+1/3}$ 和 $C_{ijk}^{n+2/3}$ 可得下述等价的差分方程:

$$\begin{aligned} & \phi_{ijk} \frac{C_{ijk}^{n+1} - C_{ijk}^n}{\Delta t} - \left\{ \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} (D \delta_{x_1} C^{n+1})_{ijk} \right. \\ & \quad \left. + \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} (D \delta_{x_2} C^{n+1})_{ijk} + \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D \delta_{x_3} C^{n+1})_{ijk} \right\} \\ & = - \sum_{\beta=1}^3 \delta_{U_{\beta}^{n+1}, x_{\beta}} C_{ijk}^{n+1} - b(C_{ijk}^n) \frac{P_{ijk}^{n+1} - P_{ijk}^n}{\Delta t} + g(X_{ijk}, t^n, C_{ijk}^n) \\ & \quad - (\Delta t)^2 \left\{ \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D \delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} (D \delta_{x_2} d_t C^n) \right] \right)_{ijk} \right. \\ & \quad + \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D \delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D \delta_{x_3} d_t C^n) \right] \right)_{ijk} \\ & \quad \left. + \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} \left(D \delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D \delta_{x_3} d_t C^n) \right] \right)_{ijk} \right\} \\ & \quad + (\Delta t)^3 \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D \delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \right. \\ & \quad \left. \left. \times \delta_{\bar{x}_2} \left(D \delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D \delta_{x_3} d_t C^n) \right] \right) \right] \right)_{ijk} + \Delta t \left\{ \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \end{aligned}$$

$$\begin{aligned} & \times \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_\beta^{n+1}, x_\beta} C^{n+1} \right) \right)_{ijk} + \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_\beta^{n+1}, x_\beta} \right. \right. \\ & \left. \left. \times C^{n+1} \right) \right)_{ijk} \Big\} - (\Delta t)^2 \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \right. \\ & \left. \left. \times \delta_{\bar{x}_2} \left(D\delta_{x_2} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_\beta^{n+1}, x_\beta} C^{n+1} \right) \right) \right) \right)_{ijk}, \quad 1 \leq i, j, k \leq N-1, \end{aligned} \quad (3.14a)$$

$$C_{ijk}^{n+1} = h_{ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad (3.14b)$$

由方程 (1.2) ($t = t^{n+1}$) 和 (3.14) 可导出饱和度函数的误差方程为

$$\begin{aligned} & \phi_{ijk} \frac{\xi_{ijk}^{n+1} - \xi_{ijk}^n}{\Delta t} - \left\{ \left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} (D\delta_{x_1} \xi^{n+1})_{ijk} + \left(1 + \frac{h}{2} |u_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \\ & \left. \times \delta_{\bar{x}_2} (D\delta_{x_2} \xi^{n+1})_{ijk} + \left(1 + \frac{h}{2} |u_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} \xi^{n+1})_{ijk} \right\} \\ & = \sum_{\beta=1}^3 \{ \delta_{U_\beta^{n+1}, x_\beta} C_{ijk}^{n+1} - \delta_{u_\beta^{n+1}, x_\beta} c_{ijk}^{n+1} \} + \sum_{\beta=1}^3 \left[\left(1 + \frac{h}{2} |u_\beta^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \\ & \left. - \left(1 + \frac{h}{2} |U_\beta^{n+1}| D^{-1} \right)_{ijk}^{-1} \right] \delta_{\bar{x}_\beta} (D\delta_{x_\beta} C^{n+1})_{ijk} + g(X_{ijk}, t^{n+1}, c_{ijk}^{n+1}) \\ & - g(X_{ijk}, t^n, C_{ijk}^n) - b(C_{ijk}^n) \frac{\pi_{ijk}^{n+1} - \pi_{ijk}^n}{\Delta t} - [b(c_{ijk}^{n+1}) - b(C_{ijk}^n)] \frac{p_{ijk}^{n+1} - p_{ijk}^n}{\Delta t} \\ & - (\Delta t)^2 \left\{ \left[\left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |u_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} (D\delta_{x_2} d_t c^n) \right] \right) \right]_{ijk} \right. \\ & \left. - \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} (D\delta_{x_2} d_t C^n) \right] \right) \right]_{ijk} \right. \\ & + \cdots + \left[\left(1 + \frac{h}{2} |u_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |u_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t c^n) \right] \right) \right]_{ijk} \\ & \left. - \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t C^n) \right] \right) \right]_{ijk} \right\} \\ & + (\Delta t)^3 \left\{ \left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |u_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \right. \right. \\ & \left. \left. \times \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |u_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t c^n) \right] \right) \right] \right)_{ijk} - \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \\ & \left. \times \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t C^n) \right] \right) \right] \right)_{ijk} \right\} \\ & + \Delta t \left\{ \left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{u_\beta^{n+1}, x_\beta} c^{n+1} \right) \right)_{ijk} - \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)_{ijk}^{-1} \right. \\ & \left. \times \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_\beta^{n+1}, x_\beta} C^{n+1} \right) \right)_{ijk} + \cdots \right\} - (\Delta t)^2 \left\{ \left(1 + \frac{h}{2} |u_1^n| D^{-1} \right)_{ijk}^{-1} \right. \end{aligned}$$

$$\begin{aligned}
 & \times \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2} |u_2^n| D^{-1} \right)^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{u_{\beta}^{n+1}, x_{\beta}} c^{n+1} \right) \right) \right) \right)_{ijk} \\
 & - \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \right. \right. \\
 & \left. \left. \times \delta_{\bar{x}_2} \left(D\delta_{x_2} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_{\beta}^{n+1}, x_{\beta}} C^{n+1} \right) \right) \right) \right)_{ijk} \Big\} + \varepsilon_{ijk}^{n+1}, \quad 1 \leq i, j, k \leq N-1,
 \end{aligned} \tag{3.15a}$$

$$\xi_{ijk}^{n+1} = 0, \quad X_{ijk} \in \partial\Omega_h, \tag{3.15b}$$

在这里由于

$$\begin{aligned}
 \frac{\partial}{\partial x_1} \left(D \frac{\partial c^{n+1}}{\partial x_1} \right)_{ijk} - \left(1 + \frac{h}{2} |u_{1,ijk}^{n+1}| D_{ijk}^{-1} \right)^{-1} \delta_{\bar{x}_1} (D\delta_{x_1} c^{n+1})_{ijk} &= \frac{h}{2} |u_{1,ijk}^{n+1}| D_{ijk}^{-1} \delta_{\bar{x}_1} (D\delta_{x_1} c^{n+1})_{ijk} + O(h^2), \\
 \frac{\partial}{\partial x_2} \left(D \frac{\partial c^{n+1}}{\partial x_2} \right)_{ijk} - \left(1 + \frac{h}{2} |u_{2,ijk}^{n+1}| D_{ijk}^{-1} \right)^{-1} \delta_{\bar{x}_2} (D\delta_{x_2} c^{n+1})_{ijk} &= \frac{h}{2} |u_{2,ijk}^{n+1}| D_{ijk}^{-1} \delta_{\bar{x}_2} (D\delta_{x_2} c^{n+1})_{ijk} + O(h^2), \\
 \frac{\partial}{\partial x_3} \left(D \frac{\partial c^{n+1}}{\partial x_3} \right)_{ijk} - \left(1 + \frac{h}{2} |u_{3,ijk}^{n+1}| D_{ijk}^{-1} \right)^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} c^{n+1})_{ijk} &= \frac{h}{2} |u_{3,ijk}^{n+1}| D_{ijk}^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} c^{n+1})_{ijk} + O(h^2),
 \end{aligned}$$

由此推得

$$|\varepsilon_{ijk}^{n+1}| \leq M \left(\left\| \frac{\partial^2 c}{\partial t^2} \right\|_{L^\infty(L^\infty)}, \left\| \frac{\partial c}{\partial t} \right\|_{L^\infty(W^{4,\infty})}, \|c\|_{L^\infty(W^{4,\infty})}, \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(W^{4,\infty})} \right) \{h^2 + \Delta t\}.$$

对饱和度误差方程 (3.15) 乘以 $\delta_t \xi_{ijk}^n = \xi_{ijk}^{n+1} - \xi_{ijk}^n = d_t \xi_{ijk}^n \Delta t$ 作内积, 并分部求和得到

$$\begin{aligned}
 & \langle \phi d_t \xi^n, d_t \xi^n \rangle \Delta t + \sum_{\beta=1}^3 \left\langle D\delta_{x_{\beta}} \xi^{n+1}, \delta_{x_{\beta}} \left[\left(1 + \frac{h}{2} |u_{\beta}^{n+1}| D^{-1} \right)^{-1} (\xi^{n+1} - \xi^n) \right] \right\rangle \\
 &= \sum_{\beta=1}^3 \langle \delta_{U_{\beta}^{n+1}, x_{\beta}} C^{n+1} - \delta_{u_{\beta}^{n+1}, x_{\beta}} c^{n+1}, d_t \xi^n \rangle \Delta t + \sum_{\beta=1}^3 \left\langle \left[\left(1 + \frac{h}{2} |u_{\beta}^{n+1}| D^{-1} \right)^{-1} \right. \right. \\
 & \quad \left. \left. - \left(1 + \frac{h}{2} |U_{\beta}^{n+1}| D^{-1} \right)^{-1} \right] \delta_{\bar{x}_{\beta}} (D\delta_{x_{\beta}} C^{n+1}), d_t \xi^n \right\rangle \Delta t + \langle g(c^{n+1}) - g(C^n), d_t \xi^n \rangle \Delta t \\
 & \quad - \left\langle b(C^n) \frac{\pi^{n+1} - \pi^n}{\Delta t}, d_t \xi^n \right\rangle \Delta t - \left\langle [b(c^{n+1}) - b(C^n)] \frac{p^{n+1} - p^n}{\Delta t}, d_t \xi^n \right\rangle \Delta t \\
 & \quad - (\Delta t)^3 \left\{ \left\langle \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_2} (D\delta_{x_2} d_t \xi^n) \right] \right), d_t \xi^n \right\rangle \right. \\
 & \quad + \left\langle \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t \xi^n) \right] \right), d_t \xi^n \right\rangle \\
 & \quad + \left\langle \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t \xi^n) \right] \right), d_t \xi^n \right\rangle + \dots \Big\} \\
 & \quad + (\Delta t)^4 \left\{ \left\langle \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \right. \right. \right. \right. \\
 & \quad \left. \left. \left. \times \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t \xi^n) \right] \right) \right] \right), d_t \xi^n \right\rangle + \dots \right\} \\
 & \quad + (\Delta t)^2 \left\{ \left\langle \left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{u_{\beta}^{n+1}, x_{\beta}} c^{n+1} \right) \right) \right. \right.
 \end{aligned}$$

首先估计上式左端第二项,

由于

于是有

类似地有

现估计 (3.16) 右端诸项. 由归纳法假定 (3.5) 可以推出 $\mathbf{U}^{n+1}$ 是有界的, 故有

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$$\begin{aligned}
 & - \left\{ \frac{U_{1,ijk}^{n+1} + |U_{1,ijk}^{n+1}|}{2} D_{ijk}^{-1} D_{i-1/2,jk} \delta_{\bar{x}_1} \xi_{ijk}^{n+1} + \frac{U_{1,ijk}^{n+1} - |U_{1,ijk}^{n+1}|}{2} D_{ijk}^{-1} D_{i+1/2,jk} \delta_{x_1} \xi_{ijk}^{n+1} \right\} + O(\Delta t) \\
 & \leq M \{ |U_{1,ijk}^{n+1} - u_{1,ijk}^{n+1}| + |\delta_{\bar{x}_1} \xi_{ijk}^{n+1}| + |\delta_{x_1} \xi_{ijk}^{n+1}| + \Delta t \}.
 \end{aligned}$$

由此可得

$$\langle \delta_{U_1^{n+1}, x_1} C^{n+1} - \delta_{u_1^{n+1}, x_1} c^{n+1}, d_t \xi^n \rangle \Delta t \leq M \{ \|u_1^{n+1} - U_1^{n+1}\|^2 + \|\delta_{x_1} \xi^{n+1}\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t.$$

类似地估计, 我们可得

$$\begin{aligned}
 & \sum_{\beta=1}^3 \langle \delta_{U_\beta^{n+1}, x_\beta} C^{n+1} - \delta_{u_\beta^{n+1}, x_\beta} c^{n+1}, d_t \xi^n \rangle \Delta t \\
 & \leq M \{ \|\mathbf{u}^{n+1} - \mathbf{U}^{n+1}\|^2 + \|\nabla_h \xi^{n+1}\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t.
 \end{aligned} \tag{3.18}$$

现估计 (3.16) 右端第二项, 注意到

$$\begin{aligned}
 & \left(1 + \frac{h}{2} |u_{\beta,ijk}^{n+1}| D_{ijk}^{-1} \right)^{-1} - \left(1 + \frac{h}{2} |U_{\beta,ijk}^{n+1}| D_{ijk}^{-1} \right)^{-1} = \frac{\frac{h}{2} (|U_{\beta,ijk}^{n+1}| - |u_{\beta,ijk}^{n+1}|) D_{ijk}^{-1}}{(1 + \frac{h}{2} |u_{\beta,ijk}^{n+1}| D_{ijk}^{-1})(1 + \frac{h}{2} |U_{\beta,ijk}^{n+1}| D_{ijk}^{-1})}, \\
 & \beta = 1, 2, 3,
 \end{aligned}$$

归纳法假定 (3.5) 有

$$\begin{aligned}
 & \sum_{\beta=1}^3 \left\langle \left[\left(1 + \frac{h}{2} |u_\beta^{n+1}| D^{-1} \right)^{-1} - \left(1 + \frac{h}{2} |U_\beta^{n+1}| D^{-1} \right)^{-1} \right] \delta_{\bar{x}_\beta} (D \delta_{x_\beta} C^{n+1}), d_t \xi^n \right\rangle \Delta t \\
 & \leq M \{ \|\mathbf{u}^{n+1} - \mathbf{U}^{n+1}\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t.
 \end{aligned} \tag{3.19}$$

对估计式 (3.16) 右端第三至五和最后一项, 由 ε_0 -Lipschitz 条件和归纳法假定 (3.5) 可以推得

$$\langle g(c^{n+1}) - g(C^n), d_t \xi^n \rangle \Delta t \leq M \{ \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t, \tag{3.20a}$$

$$- \left\langle b(C^n) \frac{\pi^{n+1} - \pi^n}{\Delta t}, d_t \xi^n \right\rangle \Delta t \leq M \|d_t \pi^n\|^2 \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t, \tag{3.20b}$$

$$- \left\langle [b(c^{n+1}) - b(C^n)] \frac{p^{n+1} - p^n}{\Delta t}, d_t \xi^n \right\rangle \Delta t \leq M \{ \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t, \tag{3.20c}$$

$$\langle \varepsilon^{n+1}, d_t \xi^n \rangle \Delta t \leq M \{ h^4 + (\Delta t)^2 \} \Delta t + \varepsilon \|d_t \xi^n\|^2 \Delta t. \tag{3.20d}$$

现在对估计式 (3.16) 右端第六项进行估计,

$$\begin{aligned}
 & - (\Delta t)^3 \left\langle \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left[D \delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_2} (D \delta_{x_2} d_t \xi^n) \right] \right], d_t \xi^n \right\rangle \\
 & = (\Delta t)^3 \left\langle D \delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_2} (D \delta_{x_2} d_t \xi^n) \right], \delta_{x_1} \left[\left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} d_t \xi^n \right] \right\rangle \\
 & = (\Delta t)^3 \left\langle D \left(\delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \right) \cdot \delta_{\bar{x}_2} (D \delta_{x_2} d_t \xi^n) + \phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \right. \right. \\
 & \quad \left. \left. \times \delta_{x_1} \delta_{\bar{x}_2} (D \delta_{x_2} d_t \xi^n) \right), \delta_{x_1} \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \cdot d_t \xi^n + \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \cdot \delta_{x_1} d_t \xi^n \right\rangle
 \end{aligned}$$

$$\begin{aligned}
&= -(\Delta t)^3 \left\{ \left\langle D\delta_{x_1}\delta_{x_2}d_t\xi^n + \delta_{x_1}D \cdot \delta_{x_2}d_t\xi^n, D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \right. \right. \\
&\quad \times \delta_{x_1}\delta_{x_2}d_t\xi^n + \left\{ \delta_{x_2}\left[D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\right] \cdot \delta_{x_1}d_t\xi^n \right. \\
&\quad + D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1} \cdot \delta_{x_1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \cdot \delta_{x_2}d_t\xi^n \\
&\quad \left. \left. + \delta_{x_2}\left[D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\delta_{x_1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\right] \cdot d_t\xi^n \right\} \right\rangle \\
&\quad + \left\langle D\delta_{x_2}d_t\xi^n, D\delta_{x_1}\left(\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1} \cdot \left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\right)\delta_{x_1}\delta_{x_2}d_t\xi^n \right. \\
&\quad + \left\{ \delta_{x_2}\left[D\delta_{x_1}\left(\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\right) \cdot \left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\right] \right. \\
&\quad \times \delta_{x_1}d_t\xi^n + D\delta_{x_1}\left(\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\right)\delta_{x_1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\delta_{x_2}d_t\xi^n \\
&\quad \left. \left. + \delta_{x_2}\left[D\delta_{x_1}\left(\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\right) \cdot \delta_{x_1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\right] \cdot d_t\xi^n \right\} \right\rangle \right\}. \quad (3.21)
\end{aligned}$$

对上式依次讨论下述诸项:

$$\begin{aligned}
&-(\Delta t)^3 \left\langle D\delta_{x_1}\delta_{x_2}d_t\xi^n, D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \cdot \delta_{x_1}\delta_{x_2}d_t\xi^n \right\rangle \\
&= -(\Delta t)^3 \sum_{i,j,k=1}^N D_{i,j-1/2,k}D_{i-1/2,j,k}\phi_{ijk}^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}_{ijk}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}_{ijk} \\
&\quad \times (\delta_{x_1}\delta_{x_2}d_t\xi_{ijk}^n)^2 h^3.
\end{aligned}$$

由于 $0 < D_* \leq D(X) \leq D^*$, $0 < \phi_* \leq \phi(X) \leq \phi^*$, 由归纳法假定推出 $\mathbf{U}^{n+1}$ 是有界的, 由此可以推出

$$0 < b_1 \leq \left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}_{ijk}, \quad 0 < b_2 \leq \left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}_{ijk}.$$

于是有

$$\begin{aligned}
&-(\Delta t)^3 \left\langle D\delta_{x_1}\delta_{x_2}d_t\xi^n, D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \cdot \delta_{x_1}\delta_{x_2}d_t\xi^n \right\rangle \\
&\leq -(\Delta t)^3 D_*^2(\phi^*)^{-1}b_1b_2 \sum_{i,j,k=1}^N (\delta_{x_1}\delta_{x_2}d_t\xi_{ijk}^n)^2 h^3. \quad (3.22a)
\end{aligned}$$

对于含有 $\delta_{x_1}\delta_{x_2}d_t\xi^n$ 的其余诸项, 它们是

$$\begin{aligned}
&-(\Delta t)^3 \left\{ \left\langle D\delta_{x_1}\delta_{x_2}d_t\xi^n, \delta_{x_2}\left[D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1}\right] \cdot \delta_{x_1}d_t\xi^n \right. \right. \\
&\quad + D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\delta_{x_1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \cdot \delta_{x_2}d_t\xi^n \left. \right\rangle \\
&\quad + \left\langle \delta_{x_1}D \cdot \delta_{x_2}d_t\xi^n, D\phi^{-1}\left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1}\left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \cdot \delta_{x_1}\delta_{x_2}d_t\xi^n \right\rangle
\end{aligned}$$

$$+ \left\langle D\delta_{x_2} d_t \xi^n, D\delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \right) \cdot \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \cdot \delta_{x_1} \delta_{x_2} d_t \xi^n \right\rangle \}. \quad (3.22b)$$

首先讨论第一项,

$$\begin{aligned} & -(\Delta t)^3 \left\langle D\delta_{x_1} \delta_{x_2} d_t \xi^n, \delta_{x_2} \left[D\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \right] \cdot \delta_{x_1} d_t \xi^n \right\rangle \\ & = -(\Delta t)^3 \sum_{i,j,k=1}^N D_{i,j-1/2,k} \delta_{x_2} \left[D_{i-1/2,jk} \phi_{ijk}^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \right]_{ijk} \\ & \quad \times \delta_{x_1} \delta_{x_2} d_t \xi_{ijk}^n \cdot \delta_{x_1} d_t \xi_{ijk}^n h^3. \end{aligned}$$

由归纳法假定和逆定理, 可以推出 $h\|\mathbf{U}^{n+1}\|_{1,\infty}$ 是有界的, 于是可推出

$$\delta_{x_2} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1}_{ijk}, \quad \delta_{x_2} \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1}_{ijk}$$

是有界的. 应用 ε -Cauchy 不等式可以推出

$$\begin{aligned} & -(\Delta t)^3 \left\langle D\delta_{x_1} \delta_{x_2} d_t \xi^n, \delta_{x_2} \left[D\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \right] \cdot \delta_{x_1} d_t \xi^n \right\rangle \\ & \leq \varepsilon (\Delta t)^3 \sum_{i,j,k=1}^N (\delta_{x_1} \delta_{x_2} d_t \xi_{ijk}^n)^2 h^3 + M (\Delta t)^3 \sum_{i,j,k=1}^N (\delta_{x_1} d_t \xi^n)^2 h^3. \end{aligned} \quad (3.22c)$$

对 (3.22b) 中其余诸项可进行类似的估算,

$$\begin{aligned} & -(\Delta t)^3 \left\{ \left\langle \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \delta_{x_2} (D\delta_{x_2} d_t \xi^n) \right] \right), d_t \xi^n \right\rangle + \cdots \right\} \\ & \leq M \{ \|\nabla_h \xi^{n+1}\|^2 + \|\nabla_h \xi^n\|^2 + \|\xi^{n+1}\|^2 + \|\xi^n\|^2 \} \Delta t. \end{aligned} \quad (3.23)$$

同理对 (3.16) 第六项其他部分, 亦可得估计式 (3.23).

对于第七项, 由限制性条件 (3.1)、归纳法假定 (3.5) 和逆估计可得

$$\begin{aligned} & (\Delta t)^4 \left\{ \left\langle \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_2^{n+1}| D^{-1} \right)^{-1} \right. \right. \right. \right. \\ & \quad \times \delta_{\bar{x}_2} \left(D\delta_{x_2} \left[\phi^{-1} \left(1 + \frac{h}{2} |U_3^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_3} (D\delta_{x_3} d_t \xi^n) \right] \right) \right], d_t \xi^n \right\rangle + \cdots \right\} \\ & \leq \varepsilon \|d_t \xi^n\|^2 \Delta t + M \{ \|\nabla_h \xi^{n+1}\|^2 + \|\nabla_h \xi^n\|^2 + \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t. \end{aligned} \quad (3.24)$$

对第八和九项可类似估计为

$$\begin{aligned} & (\Delta t)^2 \left\{ \left\langle \left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{u_{\beta}^{n+1}, x_{\beta}} c^{n+1} \right) \right) - \left(1 + \frac{h}{2} |U_1^{n+1}| D^{-1} \right)^{-1} \right. \right. \\ & \quad \times \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_{\beta}^{n+1}, x_{\beta}} C^{n+1} \right) \right), d_t \xi^n \right\rangle + \cdots \right\} - (\Delta t)^3 \left\langle \left(1 + \frac{h}{2} |u_1^{n+1}| D^{-1} \right)^{-1} \right. \\ & \quad \times \delta_{\bar{x}_1} \left(D\delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2} |u_2^{n+1}| D^{-1} \right)^{-1} \delta_{\bar{x}_2} \left(D\delta_{x_2} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{u_{\beta}^{n+1}, x_{\beta}} c^{n+1} \right) \right) \right) \right), \end{aligned}$$

$$\begin{aligned}
& - \left(1 + \frac{h}{2}|U_1^{n+1}|D^{-1}\right)^{-1} \delta_{\bar{x}_1} \left(D \delta_{x_1} \left(\phi^{-1} \left(1 + \frac{h}{2}|U_2^{n+1}|D^{-1}\right)^{-1} \right. \right. \\
& \times \left. \left. \delta_{\bar{x}_2} \left(D \delta_{x_2} \left(\phi^{-1} \sum_{\beta=1}^3 \delta_{U_\beta^{n+1}, x_\beta} C^{n+1} \right) \right) \right) \right), d_t \xi^n \rangle \\
& \leq \varepsilon \|d_t \xi^n\|^2 \Delta t + M \{ \|\mathbf{u}^{n+1} - \mathbf{U}^{n+1}\|^2 + \|\nabla_h \xi^{n+1}\|^2 + \|\nabla_h \xi^n\|^2 + \|\xi^{n+1}\|^2 \\
& \quad + \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t.
\end{aligned} \tag{3.25}$$

对误差估计式 (3.16), 应用 (3.23)–(3.25), 经计算可得

$$\begin{aligned}
& \|d_t \xi^n\|^2 \Delta t + \frac{1}{2} \sum_{\beta=1}^3 \left[\left\langle D \delta_{x_\beta} \xi^{n+1}, \left(1 + \frac{h}{2}|u_\beta^{n+1}|D^{-1}\right)^{-1} \delta_{x_\beta} \xi^{n+1} \right\rangle \right. \\
& \quad \left. - \left\langle D \delta_{x_\beta} \xi^n, \left(1 + \frac{h}{2}|u_\beta^{n+1}|D^{-1}\right)^{-1} \delta_{x_\beta} \xi^n \right\rangle \right] \\
& \leq \varepsilon \|d_t \xi^n\|^2 \Delta t + M \{ \|\mathbf{u}^{n+1} - \mathbf{U}^{n+1}\|^2 + \|d_t \pi^n\|^2 + \|\nabla_h \xi^{n+1}\|^2 \\
& \quad + \|\nabla_h \xi^n\|^2 + \|\xi^{n+1}\|^2 + \|\xi^n\|^2 + (\Delta t)^2 \} \Delta t.
\end{aligned} \tag{3.26}$$

对压力函数误差估计式 (3.13) 关于 t 求和 $0 \leq n \leq L$, 注意到 $\pi^0 = 0$, 可得

$$\begin{aligned}
& \sum_{n=0}^L \|d_t \pi^n\|^2 \Delta t + \langle A^L \nabla_h \pi^{L+1}, \nabla_h \pi^{L+1} \rangle - \langle A^0 \nabla_h \pi^0, \nabla_h \pi^0 \rangle \\
& \leq \sum_{n=1}^L \langle [A^n - A^{n-1}] \nabla_h \pi^n, \nabla_h \pi^n \rangle + M \sum_{n=1}^L \{ \|\xi^n\|_1^2 + h^4 + (\Delta t)^2 \} \Delta t.
\end{aligned} \tag{3.27}$$

对上式右端第一项的系数有

$$\begin{aligned}
A^n - A^{n-1} &= a(X, C^n) - a(X, C^{n-1}) = \frac{\partial \bar{a}}{\partial c} (C^n - C^{n-1}) = \frac{\partial \bar{a}}{\partial c} \{ (\xi^n - \xi^{n-1}) - (c^n - c^{n-1}) \} \\
&= \frac{\partial \bar{a}}{\partial t} \left\{ d_t \xi^{n-1} - \frac{\partial \bar{c}}{\partial t} \right\} \Delta t.
\end{aligned}$$

由于 $\frac{\partial \bar{a}}{\partial c}$ 和 $\frac{\partial \bar{c}}{\partial t}$ 有界, 于是有

$$|A^n - A^{n-1}| \leq M \{ |d_t \xi^{n-1}| + 1 \} \Delta t.$$

则对 (3.27) 右端第一项有下述估计:

$$\sum_{n=1}^L \langle [A^n - A^{n-1}] \nabla_h \pi^n, \nabla_h \pi^n \rangle \leq \varepsilon \sum_{n=1}^L \|d_t \xi^{n-1}\|^2 \Delta t + M \sum_{n=1}^L \|\nabla_h \pi^n\|^2 \Delta t. \tag{3.28}$$

对估计式 (3.27) 应用 (3.28) 可得

$$\begin{aligned}
& \sum_{n=0}^L \|d_t \pi^n\|^2 \Delta t + \langle A^L \nabla_h \pi^{L+1}, \nabla_h \pi^{L+1} \rangle \\
& \leq \varepsilon \sum_{n=1}^L \|d_t \xi^{n-1}\|^2 \Delta t + M \sum_{n=1}^L \{ \|\xi^n\|_1^2 + \|\nabla_h \pi^n\|^2 + h^4 + (\Delta t)^2 \} \Delta t.
\end{aligned} \tag{3.29}$$

其次将对饱和度函数的误差估计式 (3.26) 对于 t 求和 ($0 \leq n \leq L$), 注意到 $\xi^0 = 0$, 并利用估计式 $\|\mathbf{u}^{n+1} - \mathbf{U}^{n+1}\|^2 \leq M\{\|\xi^n\|^2 + \|\nabla_h \pi^{n+1}\|^2 + h^4\}$ 可得

$$\begin{aligned} & \sum_{n=0}^L \|d_t \xi^n\|^2 \Delta t + \frac{1}{2} \sum_{\beta=1}^3 \left[\left\langle D \delta_{x_\beta} \xi^{L+1}, \left(1 + \frac{h}{2} |u_\beta^{L+1}| D^{-1}\right)^{-1} \delta_{x_\beta} \xi^{L+1} \right\rangle \right. \\ & \quad \left. - \left\langle D \delta_{x_\beta} \xi^0, \left(1 + \frac{h}{2} |u_\beta^0| D^{-1}\right)^{-1} \delta_{x_\beta} \xi^0 \right\rangle \right] \\ & \leq \sum_{n=1}^L \left\{ \sum_{\beta=1}^3 \left\langle D \delta_{x_\beta} \xi^n, \left[\left(1 + \frac{h}{2} |u_\beta^{n+1}| D^{-1}\right)^{-1} - \left(1 + \frac{h}{2} |u_\beta^n| D^{-1}\right)^{-1} \right] \delta_{x_\beta} \xi^n \right\rangle \right\} \\ & \quad + \varepsilon \|d_t \xi^n\|^2 \Delta t + M\{\|d_t \pi^n\|^2 + \|\nabla_h \xi^{n+1}\|^2 + \|\nabla_h \xi^n\|^2 + \|\xi^{n+1}\|^2 + \|\xi^n\|^2 + (\Delta t)^2\} \Delta t. \quad (3.30) \end{aligned}$$

注意到

$$\begin{aligned} & \left| \left(1 + \frac{h}{2} |u_{\beta,ijk}^{n+1}| D_{ijk}^{-1}\right)^{-1} - \left(1 + \frac{h}{2} |u_{\beta,ijk}^n| D_{ijk}^{-1}\right)^{-1} \right| \\ & = \frac{\frac{h}{2} (|u_{\beta,ijk}^n| - |u_{\beta,ijk}^{n+1}|) D_{ijk}^{-1}}{(1 + \frac{h}{2} |u_{\beta,ijk}^{n+1}| D_{ijk}^{-1})(1 + \frac{h}{2} |u_{\beta,ijk}^n| D_{ijk}^{-1})} \\ & \leq \frac{\frac{h}{2} D_{ijk}^{-1} |d_t u_{\beta,ijk}^n| \Delta t}{(1 + \frac{h}{2} |u_{\beta,ijk}^{n+1}| D_{ijk}^{-1})(1 + \frac{h}{2} |u_{\beta,ijk}^n| D_{ijk}^{-1})} \leq M h \Delta t, \quad \beta = 1, 2, 3, \quad (3.31) \end{aligned}$$

对误差方程 (3.30), 应用 (3.31) 有

$$\begin{aligned} & \sum_{n=0}^L \|d_t \xi^n\|^2 \Delta t + \frac{1}{2} \sum_{\beta=1}^3 \left\langle D \delta_{x_\beta} \xi^{L+1}, \left(1 + \frac{h}{2} |u_\beta^{L+1}| D^{-1}\right)^{-1} \delta_{x_\beta} \xi^{L+1} \right\rangle \\ & \leq M \sum_{n=0}^L \|d_t \pi^n\|^2 \Delta t + M \sum_{n=0}^L \{\|\nabla_h \pi^{n+1}\|^2 + \|\nabla_h \xi^{n+1}\|_1^2 + h^4 + (\Delta t)^2\} \Delta t. \quad (3.32) \end{aligned}$$

注意到当 $\pi^0 = \xi^0 = 0$ 时有

$$\|\pi^{L+1}\|^2 \leq \varepsilon \sum_{n=0}^L \|d_t \pi^n\|^2 \Delta t + M \sum_{n=0}^L \|\pi^n\|^2 \Delta t, \quad \|\xi^{L+1}\|^2 \leq \varepsilon \sum_{n=0}^L \|d_t \xi^n\|^2 \Delta t + M \sum_{n=0}^L \|\xi^n\|^2 \Delta t.$$

组合 (3.29) 和 (3.32) 可得

$$\sum_{n=0}^L [\|d_t \pi^n\|^2 + \|d_t \xi^n\|^2] \Delta t + \|\pi^{L+1}\|_1^2 + \|\xi^{L+1}\|_1^2 \leq M \sum_{n=0}^L \{\|\pi^{n+1}\|_1^2 + \|\xi^{n+1}\|_1^2 + h^4 + (\Delta t)^2\} \Delta t. \quad (3.33)$$

应用 Gronwall 引理可得

$$\sum_{n=0}^L [\|d_t \pi^n\|^2 + \|d_t \xi^n\|^2] \Delta t + \|\pi^{L+1}\|_1^2 + \|\xi^{L+1}\|_1^2 \leq M \{h^4 + (\Delta t)^2\}. \quad (3.34)$$

下面需要检验归纳法假定 (3.5). 对于 $n = 0$, 由于 $\pi^0 = \xi^0 = 0$, 故 (3.5) 是正确的. 若对任意的正整数 l , $1 \leq n \leq l$ 时 (3.5) 成立, 由 (3.34) 可得

$$\|\pi^{l+1}\|_1 + \|\xi^{l+1}\|_1 \leq M \{h^2 + \Delta t\},$$

由限制性条件 (1.6) 和逆估计有

$$\|\pi^{l+1}\|_{1,\infty} + \|\xi^{l+1}\|_{1,\infty} \leq Mh^{1/2},$$

归纳法假定 (3.5) 成立.

在此基础上, 讨论组分浓度方程的误差估计. 为此, 将 (2.6a)、(2.6c) 和 (2.6e) 消去 $S_\alpha^{n+1/3}$ 和 $S_\alpha^{n+2/3}$ 可得

$$\begin{aligned} & \phi_{ijk} C_{ijk}^{n+1} \frac{S_{\alpha,ijk}^{n+1} - S_{\alpha,ijk}^n}{\Delta t} - \sum_{\beta=1}^3 \delta_{\bar{x}\beta} (C^{n+1} \phi K_\alpha \delta_{x\beta} S_\alpha^{n+1})_{ijk} \\ &= - \sum_{\beta=1}^3 \bar{\delta}_{U_\beta^{n+1}, x_\beta} S_{\alpha,ijk}^{n+1} + Q_\alpha(S_{\alpha,ijk}^n) - S_{\alpha,ijk}^n \left(q(C_{ijk}^{n+1}) - d(C_{ijk}^{n+1}) \frac{P_{ijk}^{n+1} - P_{ijk}^n}{\Delta t} + \phi_{ijk} \frac{C_{ijk}^{n+1} - C_{ijk}^n}{\Delta t} \right) \\ & \quad - (\Delta t)^2 \{ \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} d_t S_\alpha^n))_{ijk} + \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \\ & \quad \times \delta_{\bar{x}_3} (C^{n+1} \phi K_\alpha \delta_{x_3} d_t S_\alpha^n))_{ijk} + \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (C^{n+1} \phi K_\alpha \delta_{x_3} d_t S_\alpha^n))_{ijk} \} \\ & \quad + (\Delta t)^3 \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (C^{n+1} \phi K_\alpha \delta_{x_3} d_t S_\alpha^n))))_{ijk} \\ & \quad - \Delta t \left\{ \delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \right) \right)_{ijk} + \delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \right. \right. \right. \\ & \quad \times \sum_{\beta=1}^3 \bar{\delta}_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \left. \left. \right)_{ijk} \right\} + (\Delta t)^2 \delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \right. \right. \right. \right. \\ & \quad \times \sum_{\beta=1}^3 \bar{\delta}_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \left. \left. \right)_{ijk} \right) \right) \right) \right) \right)_{ijk}, \quad 1 \leq i, j, k \leq N-1, \quad \alpha = 1, 2, \dots, n_c, \end{aligned} \quad (3.35a)$$

$$S_{\alpha,ijk}^{n+1} = h_{\alpha,ijk}^{n+1}, \quad X_{ijk} \in \partial\Omega_h, \quad \alpha = 1, 2, \dots, n_c. \quad (3.35b)$$

由方程式 (1.7) ($t = t^{n+1}$) 和 (3.35) 可导出组分浓度函数的误差方程为

$$\begin{aligned} & \phi_{ijk} C_{ijk}^{n+1} \frac{\zeta_{\alpha,ijk}^{n+1} - \zeta_{\alpha,ijk}^n}{\Delta t} - \sum_{\beta=1}^3 \delta_{\bar{x}\beta} (C^{n+1} \phi K_\alpha \delta_{x\beta} \zeta_\alpha^{n+1})_{ijk} \\ &= \left\{ \phi (C^{n+1} - C^n) \frac{\partial s_\alpha}{\partial t} \right\}_{ijk} + \sum_{\beta=1}^3 \delta_{\bar{x}\beta} ((c^{n+1} - C^{n+1}) \phi K_\alpha \delta_{x\beta} s_\alpha^{n+1})_{ijk} + \sum_{\beta=1}^3 \{ \delta_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \\ & \quad - \delta_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \}_{ijk} + Q_\alpha(c_{ijk}^{n+1}, s_{\alpha,ijk}^{n+1}) - Q_\alpha(C_{ijk}^{n+1}, S_{\alpha,ijk}^{n+1}) + \left\{ (S_\alpha^n q(C^{n+1}) - s_\alpha^n q(c^{n+1}))_{ijk} \right. \\ & \quad + \left(s_\alpha^{n+1} d(c^{n+1}) \frac{\partial p^{n+1}}{\partial t} - S_\alpha^n d(C^{n+1}) \frac{P^{n+1} - P^n}{\Delta t} \right)_{ijk} + \left(S_\alpha^n \phi \frac{C^{n+1} - C^n}{\Delta t} - s_\alpha^{n+1} \phi \frac{\partial c^{n+1}}{\partial t} \right)_{ijk} \left. \right\} \\ & \quad - (\Delta t)^2 \{ [\delta_{\bar{x}_1} (c^{n+1} \phi K_\alpha \delta_{x_1} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (c^{n+1} \phi K_\alpha \delta_{x_2} d_t s_\alpha^n))_{ijk} \\ & \quad - \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} d_t S_\alpha^n))_{ijk}] + \dots \\ & \quad + [\delta_{\bar{x}_2} (c^{n+1} \phi K_\alpha \delta_{x_2} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (c^{n+1} \phi K_\alpha \delta_{x_3} d_t s_\alpha^n))_{ijk} \\ & \quad - \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (C^{n+1} \phi K_\alpha \delta_{x_3} d_t S_\alpha^n))_{ijk}] \} \\ & \quad + (\Delta t)^3 \{ \delta_{\bar{x}_1} (c^{n+1} \phi K_\alpha \delta_{x_1} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (c^{n+1} \phi K_\alpha \delta_{x_2} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (c^{n+1} \phi K_\alpha \delta_{x_3} d_t s_\alpha^n))))_{ijk} \\ & \quad - \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (C^{n+1} \phi K_\alpha \delta_{x_3} d_t S_\alpha^n))))_{ijk} \} \end{aligned}$$

$$\begin{aligned}
 & -\Delta t \left\{ \left[\delta_{\bar{x}_1} \left(c^{n+1} \phi K_\alpha \delta_{x_1} \left((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(c^{n+1} \phi K_\alpha \delta_{x_2} \left((c^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \right) \right) \right) \right]_{ijk} \right. \\
 & - \left[\delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \right) \right) \right) \right]_{ijk} \right\} \\
 & + (\Delta t)^2 \left\{ \delta_{\bar{x}_1} \left(c^{n+1} \phi K_\alpha \delta_{x_1} \left((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(c^{n+1} \phi K_\alpha \delta_{x_2} \left((c^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \right) \right) \right) \right)_{ijk} \right. \\
 & - \delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \right) \right) \right) \right)_{ijk} \right\} \\
 & + \varepsilon_{\alpha,ijk}, \quad 1 \leq i, j, k \leq N-1, \quad \alpha = 1, 2, \dots, n_c, \tag{3.36a}
 \end{aligned}$$

$$\zeta_{\alpha,ijk}^{n+1} = 0, \quad X_{ijk} \in \partial\Omega_h, \quad \alpha = 1, 2, \dots, n_c, \tag{3.36b}$$

其中 $|\varepsilon_{\alpha,ijk}| \leq M\{h + \Delta t\}$, $\alpha = 1, 2, \dots, n_c$.

在数值分析中, 注意到油藏区域中处处存在束缚水的特性, 即有 $c(X, t) \geq c_* > 0$, 其中 c_* 是正常数. 由于我们已证明关于水相饱和度函数 $c(X, t)$ 的收敛性分析式 (3.34), 得知对适当小的 h 和 Δt 有

$$C(X, t) \geq \frac{c_*}{2}. \tag{3.37}$$

对组分浓度误差方程式 (3.36) 乘以 $\delta_t \zeta_{\alpha,ijk}^n = d_t \zeta_{\alpha,ijk}^n \Delta t = \zeta_{\alpha,ijk}^{n+1} - \zeta_{\alpha,ijk}^n$ 作内积可得

$$\begin{aligned}
 & \left\langle \phi C^{n+1} \frac{\zeta_\alpha^{n+1} - \zeta_\alpha^n}{\Delta t}, d_t \zeta_\alpha^n \right\rangle \Delta t + \sum_{\beta=1}^3 \langle C^{n+1} \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^{n+1}, \delta_{x_\beta} (\zeta_\alpha^{n+1} - \zeta_\alpha^n) \rangle \\
 & = \left\langle \phi (c^{n+1} - c^n) \frac{\partial s_\alpha}{\partial t}, d_t \zeta_\alpha^n \right\rangle \Delta t + \sum_{\beta=1}^3 \langle \delta_{\bar{x}_\beta} ((c^{n+1} - C^{n+1}) \phi K_\alpha \delta_{x_\beta} s_\alpha^{n+1}), d_t \zeta_\alpha^n \rangle \Delta t \\
 & + \sum_{\beta=1}^3 \langle \delta_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} - \delta_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1}, d_t \zeta_\alpha^n \rangle \Delta t + \langle Q_\alpha(c^{n+1}, s_\alpha^{n+1}) - Q_\alpha(C^{n+1}, S_\alpha^{n+1}), d_t \zeta_\alpha^n \rangle \Delta t \\
 & + \langle S_\alpha^n q(C^{n+1}) - s_\alpha^n q(c^{n+1}), d_t \zeta_\alpha^n \rangle \Delta t + \left\langle s_\alpha^{n+1} d(c^{n+1}) \frac{\partial p^{n+1}}{\partial t} - S_\alpha^n d(C^{n+1}) \frac{P^{n+1} - P^n}{\Delta t}, d_t \zeta_\alpha^n \right\rangle \Delta t \\
 & + \left\langle S_\alpha^n \phi \frac{C^{n+1} - C^n}{\Delta t} - s_\alpha^{n+1} \phi \frac{\partial c^{n+1}}{\partial t}, d_t \zeta_\alpha^n \right\rangle \Delta t \\
 & - (\Delta t)^3 \{ (\delta_{\bar{x}_1} (c^{n+1} \phi K_\alpha \delta_{x_1} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (c^{n+1} \phi K_\alpha \delta_{x_2} d_t s_\alpha^n))) \\
 & - \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} d_t S_\alpha^n))), d_t \zeta_\alpha^n \} + \dots \} \\
 & + (\Delta t)^4 \langle \delta_{\bar{x}_1} (c^{n+1} \phi K_\alpha \delta_{x_1} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (c^{n+1} \phi K_\alpha \delta_{x_2} ((c^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (c^{n+1} \phi K_\alpha \delta_{x_3} d_t s_\alpha^n)))) \\
 & - \delta_{\bar{x}_1} (C^{n+1} \phi K_\alpha \delta_{x_1} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} (C^{n+1} \phi K_\alpha \delta_{x_2} ((C^{n+1} \phi)^{-1} \delta_{\bar{x}_3} (C^{n+1} \phi K_\alpha \delta_{x_3} d_t S_\alpha^n))))), d_t \zeta_\alpha^n \rangle \\
 & - (\Delta t)^2 \left\langle \left[\delta_{\bar{x}_1} \left(c^{n+1} \phi K_\alpha \delta_{x_1} \left((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(c^{n+1} \phi K_\alpha \delta_{x_2} \left((c^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \right) \right) \right) \right) \right] \right. \\
 & - \left[\delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \right) \right) \right) \right) \right], d_t \zeta_\alpha^n \rangle \\
 & + (\Delta t)^3 \left\langle \delta_{\bar{x}_1} \left(c^{n+1} \phi K_\alpha \delta_{x_1} \left((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(c^{n+1} \phi K_\alpha \delta_{x_2} \left((c^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \right) \right) \right) \right) \right.
 \end{aligned}$$

$$\begin{aligned}
& -\delta_{\bar{x}_1}\left(C^{n+1}\phi K_\alpha\delta_{x_1}\left((C^{n+1}\phi)^{-1}\delta_{\bar{x}_2}\left(C^{n+1}\phi K_\alpha\delta_{x_2}\left((C^{n+1}\phi)^{-1}\sum_{\beta=1}^3\bar{\delta}_{U_\beta^{n+1},x_\beta}S_\alpha^{n+1}\right)\right)\right)\right),d_t\zeta_\alpha^n\Big\rangle \\
& +\langle\varepsilon_\alpha,d_t\zeta_\alpha^n\rangle\Delta t.
\end{aligned} \tag{3.38}$$

首先估计 (3.38) 左端诸项,

$$\langle\phi C^{n+1}d_t\zeta_\alpha^n,d_t\zeta_\alpha^n\rangle\Delta t\geq\frac{1}{2}\phi_*c_*\|d_t\zeta_\alpha^n\|^2\Delta t, \tag{3.39a}$$

$$\begin{aligned}
& \sum_{\beta=1}^3\langle C^{n+1}\phi K_\alpha\delta_{x_\beta}\zeta_\alpha^{n+1},\delta_{x_\beta}(\zeta_\alpha^{n+1}-\zeta_\alpha^n)\rangle \\
& \geq\frac{1}{2}\sum_{\beta=1}^3\{\langle C^{n+1}\phi K_\alpha\delta_{x_\beta}\zeta_\alpha^{n+1},\delta_{x_\beta}\zeta_\alpha^{n+1}\rangle-\langle C^{n+1}\phi K_\alpha\delta_{x_\beta}\zeta_\alpha^n,\delta_{x_\beta}\zeta_\alpha^n\rangle\}.
\end{aligned} \tag{3.39b}$$

再估计 (3.38) 右端诸项,

$$\left\langle\phi(c^{n+1}-c^n)\frac{\partial s_\alpha}{\partial t},d_t\zeta_\alpha^n\right\rangle\Delta t\leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{h^2+\Delta t\}\Delta t, \tag{3.40a}$$

$$\begin{aligned}
& \sum_{\beta=1}^3\langle\delta_{\bar{x}_\beta}((c^{n+1}-C^{n+1})\phi K_\alpha\delta_{x_\beta}s_\alpha^{n+1}),d_t\zeta_\alpha^n\rangle\Delta t\leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\sum_{\beta=1}^3\|\delta_{x_\beta}\xi_\alpha^{n+1}\|^2\Delta t \\
& \leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{h^4+(\Delta t)^2\}\Delta t,
\end{aligned} \tag{3.40b}$$

$$\begin{aligned}
& \sum_{\beta=1}^3\langle\delta_{U_\beta^{n+1},x_\beta}S_\alpha^{n+1}-\delta_{u_\beta^{n+1},x_\beta}s_\alpha^{n+1},d_t\zeta_\alpha^n\rangle\Delta t\leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\mathbf{u}^{n+1}-\mathbf{U}^{n+1}\|^2+\|\nabla_h\zeta_\alpha^{n+1}\|^2\}\Delta t \\
& \leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\nabla_h\zeta_\alpha^{n+1}\|^2+h^4+(\Delta t)^2\}\Delta t,
\end{aligned} \tag{3.40c}$$

$$\begin{aligned}
& \langle Q_\alpha(c^{n+1},s_\alpha^{n+1})-Q_\alpha(C^{n+1},S_\alpha^{n+1}),d_t\zeta_\alpha^n\rangle\Delta t\leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\xi_\alpha^{n+1}\|^2+\|\zeta_\alpha^{n+1}\|^2\}\Delta t \\
& \leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\zeta_\alpha^{n+1}\|^2+h^4+(\Delta t)^2\}\Delta t,
\end{aligned} \tag{3.40d}$$

$$\langle S_\alpha^n q(C^{n+1})-s_\alpha^n q(c^{n+1}),d_t\zeta_\alpha^n\rangle\Delta t\leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\zeta_\alpha^n\|^2+h^4+(\Delta t)^2\}\Delta t, \tag{3.40e}$$

$$\begin{aligned}
& \left\langle s_\alpha^{n+1}d(c^{n+1})\frac{\partial p^{n+1}}{\partial t}-S_\alpha^n d(C^{n+1})\frac{P^{n+1}-P^n}{\Delta t},d_t\zeta_\alpha^n\right\rangle\Delta t \\
& \leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\zeta_\alpha^n\|^2+h^4+(\Delta t)^2\}\Delta t,
\end{aligned} \tag{3.40f}$$

$$\left\langle S_\alpha^n\phi\frac{C^{n+1}-C^n}{\Delta t}-s_\alpha^{n+1}\phi\frac{\partial c^{n+1}}{\partial t},d_t\zeta_\alpha^n\right\rangle\Delta t\leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\zeta_\alpha^n\|^2+h^4+(\Delta t)^2\}\Delta t. \tag{3.40g}$$

对于估计式 (3.38) 右端第八项进行估计可得

$$\begin{aligned}
& -(\Delta t)^3\{\langle\delta_{\bar{x}_1}(c^{n+1}\phi K_\alpha\delta_{x_1}((c^{n+1}\phi)^{-1}\delta_{\bar{x}_2}(c^{n+1}\phi K_\alpha\delta_{x_2}d_t s_\alpha^n))) \\
& \quad -\delta_{\bar{x}_1}(C^{n+1}\phi K_\alpha\delta_{x_1}((C^{n+1}\phi)^{-1}\delta_{\bar{x}_2}(C^{n+1}\phi K_\alpha\delta_{x_2}d_t S_\alpha^n))),d_t\zeta_\alpha^n\rangle+\cdots\} \\
& \leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\xi_\alpha^{n+1}\|^2+\|\nabla_h\zeta_\alpha^{n+1}\|^2+h^4+(\Delta t)^2\}\Delta t \\
& \leq\varepsilon\|d_t\zeta_\alpha^n\|^2\Delta t+M\{\|\nabla_h\zeta_\alpha^{n+1}\|^2+h^4+(\Delta t)^2\}\Delta t.
\end{aligned} \tag{3.41a}$$

对于估计式 (3.38) 右端第九项的估计是类似的,

$$(\Delta t)^4\langle\delta_{\bar{x}_1}(c^{n+1}\phi K_\alpha\delta_{x_1}((c^{n+1}\phi)^{-1}\delta_{\bar{x}_2}(c^{n+1}\phi K_\alpha\delta_{x_2}((c^{n+1}\phi)^{-1}\delta_{\bar{x}_3}(c^{n+1}\phi K_\alpha\delta_{x_3}d_t s_\alpha^n))))))\rangle$$

$$\begin{aligned}
 & -\delta_{\bar{x}_1}(C^{n+1}\phi K_\alpha \delta_{x_1}((C^{n+1}\phi)^{-1}\delta_{\bar{x}_2}(C^{n+1}\phi K_\alpha \delta_{x_2}((C^{n+1}\phi)^{-1}\delta_{\bar{x}_3}(C^{n+1}\phi K_\alpha \delta_{x_3} d_t S_\alpha^n))))_{ijk}, d_t \zeta_\alpha^n \rangle \\
 & \leq \varepsilon \|d_t \zeta_\alpha^n\|^2 \Delta t + M\{\|\nabla_h \zeta_\alpha^{n+1}\|^2 + h^4 + (\Delta t)^2\} \Delta t.
 \end{aligned} \quad (3.41b)$$

对于估计式 (3.38) 右端第十项进行估计可得

$$\begin{aligned}
 & -(\Delta t)^2 \left\langle \left[\delta_{\bar{x}_1} \left(c^{n+1} \phi K_\alpha \delta_{x_1} \left((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(c^{n+1} \phi K_\alpha \delta_{x_2} \left((c^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \right) \right) \right) \right) \right] \right. \\
 & \quad \left. - \left[\delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \right) \right) \right) \right) \right], d_t \zeta_\alpha^n \right\rangle \\
 & \leq \varepsilon \|d_t \zeta_\alpha^n\|^2 \Delta t + M\{\|\mathbf{u}^{n+1} - \mathbf{U}^{n+1}\|^2 + \|\nabla_h \zeta_\alpha^{n+1}\|^2 + \|\xi^{n+1}\|^2\} \Delta t \\
 & \leq \varepsilon \|d_t \zeta_\alpha^n\|^2 \Delta t + M\{\|\nabla_h \zeta_\alpha^{n+1}\|^2 + h^4 + (\Delta t)^2\} \Delta t.
 \end{aligned} \quad (3.41c)$$

对于估计式 (3.38) 右端第十一项的估计是类似的,

$$\begin{aligned}
 & (\Delta t)^3 \left\langle \delta_{\bar{x}_1} \left(c^{n+1} \phi K_\alpha \delta_{x_1} \left((c^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(c^{n+1} \phi K_\alpha \delta_{x_2} \left((c^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{u_\beta^{n+1}, x_\beta} s_\alpha^{n+1} \right) \right) \right) \right) \right. \\
 & \quad \left. - \delta_{\bar{x}_1} \left(C^{n+1} \phi K_\alpha \delta_{x_1} \left((C^{n+1} \phi)^{-1} \delta_{\bar{x}_2} \left(C^{n+1} \phi K_\alpha \delta_{x_2} \left((C^{n+1} \phi)^{-1} \sum_{\beta=1}^3 \bar{\delta}_{U_\beta^{n+1}, x_\beta} S_\alpha^{n+1} \right) \right) \right) \right), d_t \zeta_\alpha^n \right\rangle \\
 & \leq \varepsilon \|d_t \zeta_\alpha^n\|^2 \Delta t + M\{\|\nabla_h \zeta_\alpha^{n+1}\|^2 + h^4 + (\Delta t)^2\} \Delta t.
 \end{aligned} \quad (3.41d)$$

对于估计式 (3.38) 最后一项有

$$\langle \varepsilon_\alpha, d_t \zeta_\alpha^n \rangle \Delta t \leq \varepsilon \|d_t \zeta_\alpha^n\|^2 \Delta t + M\{h^2 + (\Delta t)^2\} \Delta t. \quad (3.41e)$$

对估计式 (3.38) 的左、右两端分别应用 (3.39)–(3.41), 可得

$$\begin{aligned}
 & \frac{1}{2} \phi_* c_* \|d_t \zeta_\alpha^n\|^2 \Delta t + \frac{1}{2} \sum_{\beta=1}^3 \{ \langle C^{n+1} \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^{n+1}, \delta_{x_\beta} \zeta_\alpha^{n+1} \rangle - \langle C^{n+1} \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^n, \delta_{x_\beta} \zeta_\alpha^n \rangle \} \\
 & \leq \varepsilon \|d_t \zeta_\alpha^n\|^2 \Delta t + M\{\|\nabla_h \zeta_\alpha^{n+1}\|^2 + \|\zeta_\alpha^{n+1}\|^2 + h^2 + (\Delta t)^2\} \Delta t.
 \end{aligned} \quad (3.42)$$

将组分浓度函数的误差估计式 (3.38) 对 t 求和 ($0 \leq n \leq L$), 注意到 $\zeta_\alpha^0 = 0$ 可得

$$\begin{aligned}
 & \sum_{n=0}^L \|d_t \zeta_\alpha^n\|^2 \Delta t + \sum_{\beta=1}^3 \{ \langle C^{L+1} \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^{L+1}, \delta_{x_\beta} \zeta_\alpha^{L+1} \rangle - \langle C^1 \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^0, \delta_{x_\beta} \zeta_\alpha^0 \rangle \} \\
 & \leq \sum_{n=0}^L \left\{ \sum_{\beta=1}^3 \langle [C^{n+1} - C^n] \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^n, \delta_{x_\beta} \zeta_\alpha^n \rangle \right\} \\
 & \quad + M \sum_{n=0}^L \{ \|\nabla_h \zeta_\alpha^{n+1}\|^2 + \|\zeta_\alpha^{n+1}\|^2 + h^2 + (\Delta t)^2 \} \Delta t.
 \end{aligned} \quad (3.43)$$

对于上式右端第一项有下述估计:

$$\sum_{n=0}^L \left\{ \sum_{\beta=1}^3 \langle [C^{n+1} - C^n] \phi K_\alpha \delta_{x_\beta} \zeta_\alpha^n, \delta_{x_\beta} \zeta_\alpha^n \rangle \right\}$$

$$\leq \varepsilon \sum_{n=0}^L \|d_t \zeta_\alpha^n\|^2 \Delta t + M \sum_{n=0}^L \|\nabla_h \zeta_\alpha^n\|^2 \Delta t \leq M \sum_{n=0}^L \{\|\nabla_h \zeta_\alpha^n\|^2 + h^4 + (\Delta t)^2\}. \quad (3.44)$$

于是可得

$$\sum_{n=0}^L \|d_t \zeta_\alpha^n\|^2 \Delta t + \sum_{\beta=1}^3 \|\delta_{x_\beta} \zeta_\alpha^{n+1}\|^2 \leq M \sum_{n=0}^L \{\|\nabla_h \zeta_\alpha^{n+1}\|^2 + \|\zeta_\alpha^{n+1}\|^2 + h^2 + (\Delta t)^2\} \Delta t. \quad (3.45)$$

注意到 $\zeta_\alpha^0 = 0$ 时有

$$\|\zeta_\alpha^{L+1}\|^2 = \varepsilon \sum_{n=0}^L \|d_t \zeta_\alpha^n\|^2 \Delta t + M \sum_{n=0}^L \|\zeta_\alpha^n\|^2 \Delta t,$$

于是有

$$\sum_{n=0}^L \|d_t \zeta_\alpha^n\|^2 \Delta t + \|\zeta_\alpha^{L+1}\|_1^2 \leq M \sum_{n=0}^L \{\|\zeta_\alpha^{n+1}\|_1^2 + h^2 + (\Delta t)^2\} \Delta t. \quad (3.46)$$

应用 Gronwall 引理可得

$$\sum_{n=0}^L \|d_t \zeta_\alpha^n\|^2 \Delta t + \|\zeta_\alpha^{L+1}\|_1^2 \leq M \{h^2 + (\Delta t)^2\}. \quad (3.47)$$

定理证毕. $\square$

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Convergence analysis of an implicit upwind difference fractional steps method of three-dimensional enhanced oil recovery percolation coupled system

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Abstract A type of implicit upwind difference fractional steps scheme is presented for the percolation coupled system of enhanced (chemical) oil recovery simulation in this paper. An optimal-order error estimate is discussed in l^2 -norm by introducing many theories and techniques such as variation form, energy method, product exchange theory of difference operators, splitting of high-order difference operators, priori estimates method of differential equations. This numerical method has been applied successfully in enhanced oil recovery simulation in actual oilfields and gives rise to satisfying simulation results.

Keywords enhanced (chemical) oil recovery, three-dimensional percolation coupled system, implicit upwind difference fractional steps, optimal order l^2 estimates, reservoir numerical simulation

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