论文

预应力梁等效刚度与固有频率的近似解析解

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摘要 预应力梁被广泛应用于微传感器和执行器中, 其刚度、固有频率与预应力的关系是非常重要的物理关系. 针对纯弯曲问题容易得到静态变形, 而纯张力问题容易得到振动频率的特点, 基于张力不影响弯曲振型和振型不影响张紧频率的假设, 本文提出了高精度近似求解新方法, 获得了两端简支、两端固定、一端固定一端简支三种典型边界条件下的等效刚度与固有频率的高精度近似解. 与有限元分析结果相比, 最大刚度相对误差为 1.6%, 频率相对误差为 2.21%. 本文研究为预应力梁的等效刚度和频率计算提供了一种简单有效的方法.

关键词 预应力梁, 刚度, 频率, 近似解析解

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1 引言

预应力梁被广泛应用于微型传感器和执行器中, 将力的变化转换为共振频率或变形刚度的变化^[1], 用于测量压力、力(重量)、吸附的质量^[2,3]、流量^[4]、热应力^[5-7]、加速度^[8]、离心加速度(或离心载荷)^[9]和胎压监测^[10]. 因此, 静态刚度, 特别是固有频率, 与预应力(或张力)之间关系是非常重要的物理关系.

预应力梁通常有两端简支、两端固定和一端固定一端简支三种典型边界条件, 其中两端固定的预应力梁应用最为广泛^[2,5-9,11-30]. 然而, 由于预应力梁非

线性振动方程的复杂性, 只有两端简支边界条件才存在解析解, 而最广泛使用的两端固定预应力梁, 其特征方程为超越方程, 难于直接获得解析解^[1]. 因此, 很多文献讨论了两端固定梁的固有频率与张力关系的近似解, 所提的方法包含实验参数拟合法^[31,32], 有限元数值求解法^[10,13,17,23,24,33]和近似求解方法^[1,2,25-27,34,35].

近似求解方法主要有两类: 一类是通过能量法, 建立变分模型, 通过数值求解或者求解技巧, 获得近似解, 该方法求解过程复杂; 另一类是通过分别计算弯曲刚度和张力引起的刚度, 进行叠加得到等效刚

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度^[36], 再计算频率, 该方法直观, 但由于在计算张力引起刚度时, 变形被假设为直线, 与实际变形相去甚远.

通过大量研究可以发现, 梁的纯弯曲问题静态变形求解较为容易, 而固有频率求解比较困难; 与之相反, 纯张力问题的固有频率求解较为容易, 而静态变形求解困难. 另外, 三种边界条件下梁的变形基本相似, 均为单凸三次曲线, 只是振幅差别较大. 瑞利黎兹方法是架起振型(变形)与特征频率的桥梁, 并且是二阶收敛, 被用来求解两端固定预应力梁的近似解析解, 但需要利用刚度、质量密度和变形函数积分来求解势能和动能^[1], 计算比较繁琐. 因此, 本文将给定振型下的动能积分等效到受力点, 提出了一种统一的解决思路: 先求梁纯弯曲变形函数及等效弯曲刚度; 其次, 以弯曲变形为振型函数, 计算动能相等条件下的等效质量系数; 再根据张力与频率关系, 计算给定振型下由张力引起的刚度; 最后将弯曲刚度与张紧刚度叠加后, 并通过等效质量求得等效频率.

采用本文方法, 推导了两端固定、两端简支和一端固定一端简支三种典型边界条件预应力梁的等效刚度和频率解析解, 与有限元分析结果相比, 最大相对误差仅 2.21%, 证明了本文计算方法的有效性.

2 技术背景

对于受定常轴向力 N 的等截面均质直梁, 其自由振动微分方程为

$$\rho A \frac{\partial^2 w}{\partial t^2} - N \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} = 0, \quad (1)$$

其中 ρ 为线密度, E 为弹性模量, I 为弯曲惯性矩, w 为变形函数, x 为轴向坐标, t 为时间参数.

令

$$w(x, t) = W(x) \sin(\omega t + \theta),$$

则(1)式化为

$$W^{(4)}(x) - \alpha^2 W''(x) - \beta^4 W(x) = 0, \quad (2)$$

其中,

$$\alpha = \sqrt{\frac{N}{EI}}, \quad \beta^4 = \omega^2 \frac{\rho A}{EI}.$$

方程(2)的解为

$$W(x) = a_1 \cos(s_1 x) + a_2 \sin(s_1 x) + a_3 \cosh(s_2 x) + a_4 \sinh(s_2 x), \quad (3)$$

其中,

$$s_1 = \sqrt{-\frac{\alpha^2}{2} + \sqrt{\frac{\alpha^4}{4} + \beta^4}}, \quad s_2 = \sqrt{\frac{\alpha^2}{2} + \sqrt{\frac{\alpha^4}{4} + \beta^4}}. \quad (4)$$

下面我们以两端固定、中间受集中载荷的预应力梁为例, 说明现有的求解方法.

此时边界条件为

$$W(0) = 0, W(L) = 0, W'(0) = 0, W'(L) = 0, \quad (5)$$

代入(3)式, 可得

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \cos(s_1 L) & \sin(s_1 L) & \cosh(s_2 L) & \sinh(s_2 L) \\ 0 & s_1 & 0 & s_2 \\ -s_1 \sin(s_1 L) & s_1 \cos(s_1 L) & s_2 \sinh(s_2 L) & s_2 \cosh(s_2 L) \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (6)$$

令系数行列式为 0, 可得特征方程为

$$2s_1 s_2 (\cos(s_1 L) \cosh(s_2 L) - 1) + (s_1^2 - s_2^2) \sin(s_1 L) \sinh(s_2 L) = 0. \quad (7)$$

由特征方程(7)的表达式可知, 该特征方程式为超越方程, 难于获得解析解.

2.1 瑞利黎兹法

为了获得频率的近似解, 目前广泛采用的是瑞利黎兹法^[1], 步骤如下.

假设振型为

$$w(x, t) = \phi(x) e^{i\omega t}, \quad (8)$$

采用瑞利黎兹法, 频率求解如下

$$\omega_n \approx \omega_n(0) \sqrt{1 + \gamma_n \frac{NL^2}{12EI}}, \quad (9)$$

其中

$$\omega_n(0) = \frac{a_n^2}{L^2} \sqrt{\frac{EI}{\rho A}}, \quad (10)$$

系数 a_n 和 γ_n 分别由下式计算

$$a_n^2 = L^2 \sqrt{\left(\int_0^L \frac{d^2 \tilde{\phi}(x)}{dx^2} dx \right)^2} / \int_0^L (\tilde{\phi}(x))^2 dx, \quad (11)$$

$$\gamma_n = \frac{12}{L^2} \left(\int_0^L \frac{d \tilde{\phi}(x)}{dx^2} dx \right)^2 / \int_0^L \left(\frac{d^2 \tilde{\phi}_n(x)}{dx^2} \right)^2 dx, \quad (12)$$

其中 $\tilde{\phi}(x)$ 代表近似振型函数, 可由不考虑张力时的变形函数替代. 采用瑞利黎兹法时, 需要计算各振型的积分函数, 计算复杂, 且不直观. 此外, 等效刚度需要计算该振型下的等效质量才能获得.

2.2 直接刚度法

将预应力梁的刚度分析弯曲刚度和张力引起的刚度两部分^[36]. 首先不考虑轴向张力, 根据弯曲变形计算出等效弯曲刚度

$$k_B = F/W_F, \quad (13)$$

F 为集中载荷, W_F 为受力点的挠度.

其次, 不考虑弯曲刚度, 只考虑轴向预紧力为 N 时, 在外力作用下, 梁的挠度为 y . 假设梁的弯曲是小变形, 施加的垂直载荷与梁的内力在垂直方向的投影相等, 则有

$$y = \frac{FL}{4N}. \quad (14)$$

等效张力刚度为

$$k_T = \frac{F}{y} = \frac{4N}{L}. \quad (15)$$

直接刚度法的优点是直观, 但其小变形假设限制了其应用范围. 另外, 在计算张力引起的刚度时, 变形为直线的假设也与实际变形相差甚远. 而且, 频率也不能直接计算得到.

因此, 本文综合瑞利黎兹法和等效刚度法的优点, 提出了一种新的解法. 其基本假设如下所述.

(1) 张紧力不影响弯曲变形. 由于张紧力为轴向力, 与弯曲变形方向垂直, 自然满足该假设.

(2) 弯曲变形不影响张紧力引起的固有频率. 由于三种边界条件的变形函数属于同一线型, 均为三次曲线, 主要差别是振幅大小, 根据瑞利黎兹法的超收敛性, 振型对特征值的影响不大.

3 预应力梁等效刚度和频率的新解法

由于梁刚度可以由某一点的受力与变形的比值来衡量, 可以假设梁的长度为 L , 假设由载荷 F 引起的弯曲变形函数为 $W(x)$, 受力点对应的挠度记为 W_F .

3.1 弯曲变形引起的刚度

弯曲刚度的计算方法与直接刚度法一样, 具体见方程(13).

3.2 给定振型下的等效质量系数

将弯曲变形函数 $W(x)$ 近似为实际振型, 假设固有频率为 ω , 则受力点的速度为 ωW_F . 假设作用点的等效质量系数为 c , 根据动能相等原则, 有

$$c\rho AL[\omega W_F]^2 = \int_L \rho A[\omega W(x)]^2 dx, \quad (16)$$

式(16)化简整理后, 得到等效质量系数

$$c = \frac{\int_L [W(x)]^2 dx}{LW_F^2}. \quad (17)$$

3.3 由张力引起的刚度

张力所引起的振型受弯曲刚度的影响而发生了变化, 但由于张力是轴向力, 对弯曲变形的形状不会产生太大的影响, 因此可以将弯曲变形当作是张力引起的实际振型, 则根据张力引起振型的频率 ω_T 计算公式可得

$$\frac{k_T}{c\rho AL} = \omega_T^2 = \left(2\pi \frac{1}{2L} \sqrt{\frac{N}{\rho A}}\right)^2. \quad (18)$$

整理后, 得到张力引起的刚度为

$$k_T = \frac{c\pi^2 N}{L}. \quad (19)$$

3.4 等效刚度和固有频率

最终的等效刚度 k 由弯曲刚度与张力刚度叠加而成, 则有

$$k = k_B + k_T = \frac{F}{W_F} + \frac{c\pi^2 N}{L}, \quad (20)$$

与弯曲振型下的等效质量组成振动系统, 对应的等效固有频率为

$$f = \frac{1}{2\pi} \sqrt{\frac{k_B + k_T}{c\rho AL}} = \frac{1}{2\pi} \sqrt{\frac{k_B}{c\rho AL} + \frac{\pi^2 N}{\rho AL^2}}. \quad (21)$$

4 数值算例

为了说明本方法的有效性, 对预应力梁三种典型工况(图 1-3)进行了分析, 并将所得的近似解与有限元分析结果进行比较. 为了便于比较, 假设三种情况均为梁中点处受集中力 F , 梁的长度均为 0.050 m, 截面宽度为 0.015 m, 厚度为 0.001 m, 所用材料为钢, 弹性模量为 200 GPa, 密度为 7850 kg/m³.

图 1 两端简支梁模型示意图

Figure 1 Schematic diagram of doubly hinged beam.

图 2 两端固支梁模型示意图

Figure 2 Schematic diagram of doubly clamped beam.

图 3 一端简支一端固支梁模型示意图

Figure 3 Schematic diagram of clamped-hinged beam.

4.1 两端简支梁的等效刚度频率近似解

挠度方程为

$$W(x) = \begin{cases} \frac{F(L^3 - 6Lx^2 - 4x^3)}{48EI} & \left(-\frac{L}{2} \leq x \leq 0\right), \\ \frac{F(L^3 - 6Lx^2 + 4x^3)}{48EI} & \left(0 \leq x \leq \frac{L}{2}\right). \end{cases} \quad (22)$$

等效弯曲刚度为

$$k_B = \frac{F}{W_F} = \frac{F}{W(0)} = \frac{48EI}{L^3}. \quad (23)$$

对应的等效质量系数为

$$c = \frac{1}{LW_F^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} [W(x)]^2 dx = \left(\frac{17}{80640} \frac{F^2 L^7}{E^2 I^2} \right) / \left(L \cdot \left(\frac{1}{48} \frac{FL^3}{EI} \right)^2 \right) = \frac{17}{35}. \quad (24)$$

对应的等效张力刚度为

$$k_T = \frac{\pi^2 c N}{L} = \frac{17\pi^2 N}{35L}. \quad (25)$$

两端简支梁的等效刚度为

$$k = \frac{FL^3}{48EI} + \frac{17\pi^2 N}{35L}. \quad (26)$$

两端简支梁的等效振动频率为

$$f = \frac{1}{2\pi} \sqrt{\frac{k_B}{c\rho AL} + \frac{\pi^2 N}{\rho AL^2}} = \frac{1}{2\pi} \sqrt{\frac{48EI}{c\rho AL^4} + \frac{\pi^2 N}{\rho AL^2}} = \frac{1}{2\pi} \sqrt{\frac{1680EI}{17\rho AL^4} + \frac{\pi^2 N}{\rho AL^2}}. \quad (27)$$

4.2 两端固支梁的频率近似解

挠度方程为

$$W(x) = \begin{cases} \frac{F(L^3 - 12Lx^2 - 16x^3)}{192EI}, & \left(-\frac{L}{2} \leq x \leq 0\right), \\ \frac{F(L^3 - 12Lx^2 + 16x^3)}{192EI}, & \left(0 \leq x \leq \frac{L}{2}\right). \end{cases} \quad (28)$$

等效弯曲刚度为

$$k_B = \frac{F}{W_F} = \frac{F}{W(0)} = \frac{192EI}{L^3}. \quad (29)$$

对应的等效质量系数为

$$c = \frac{1}{LW_F^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} [W(x)]^2 dx = \left(\frac{13}{1290240} \frac{F^2 L^7}{E^2 I^2} \right) / \left(L \cdot \left(\frac{1}{192} \frac{FL^3}{EI} \right)^2 \right) = \frac{13}{35}. \quad (30)$$

对应的等效张力刚度为

$$k_T = \frac{\pi^2 c N}{L} = \frac{13\pi^2 N}{35L}. \quad (31)$$

两端固定梁的等效刚度为

$$k = \frac{192EI}{L^3} + \frac{13\pi^2 N}{35L}. \quad (32)$$

两端固定梁的等效振动频率为

$$f = \frac{1}{2\pi} \sqrt{\frac{k_B}{c\rho AL} + \frac{\pi^2 N}{\rho AL^2}} = \frac{1}{2\pi} \sqrt{\frac{192EI}{c\rho AL^4} + \frac{\pi^2 N}{\rho AL^2}} = \frac{1}{2\pi} \sqrt{\frac{6720EI}{13\rho AL^4} + \frac{\pi^2 N}{\rho AL^2}}. \quad (33)$$

4.3 一端简支, 一端固支梁的频率近似解

挠度方程为

$$W(x) = \begin{cases} \frac{F(7L^3 + 6L^2x - 60Lx^2 - 88x^3)}{768EI}, & \left(-\frac{L}{2} \leq x \leq 0\right), \\ \frac{F(7L^3 + 6L^2x - 60Lx^2 + 40x^3)}{768EI}, & \left(0 \leq x \leq \frac{L}{2}\right). \end{cases} \quad (34)$$

等效弯曲刚度为

$$k_B = \frac{F}{W_F} = \frac{F}{W(0)} = \frac{768EI}{7L^3}. \quad (35)$$

对应的等效质量系数为

$$c = \frac{1}{LW_F^2} \int_{-\frac{L}{2}}^{\frac{L}{2}} [W(x)]^2 dx$$

$$= \left(\frac{191}{5160960} \frac{F^2 L^7}{E^2 I^2} \right) / \left(L \cdot \left(\frac{7}{768} \frac{FL^3}{EI} \right)^2 \right) = \frac{764}{1715}. \quad (36)$$

对应的等效张力刚度为

$$k_T = \frac{\pi^2 cN}{L} = \frac{764\pi^2 N}{1715L}. \quad (37)$$

一端固定一端简支梁的等效刚度为

$$k = \frac{768EI}{7L^3} + \frac{764\pi^2 N}{1715L}. \quad (38)$$

一端固定一端简支梁的等效固有频率为

$$f = \frac{1}{2\pi} \sqrt{\frac{k_B}{c\rho AL} + \frac{\pi^2 N}{\rho AL^2}} = \frac{1}{2\pi} \sqrt{\frac{768EI}{7c\rho AL^4} + \frac{\pi^2 N}{\rho AL^2}}$$

$$= \frac{1}{2\pi} \sqrt{\frac{94080EI}{397\rho AL^4} + \frac{\pi^2 N}{\rho AL^2}}. \quad (39)$$

4.4 频率近似解与有限元分析计算结果对比

为了比较近似计算的精度, 在 COMSOL Multiphysics 软件中建立了对应梁模型, 统一施加 10 N 的集中力和强度为 100 MPa 的预应力, 考虑几何非线性, 采用稳态求解受力点的变形, 计算等效刚度, 并分析预应力作用下的特征频率, 结果均列于表 1 中. 从表 1 可以看到, 等效刚度最大相对误差为 1.6%, 等效频率最大相对误差为 2.21%, 均表现出很高的计算精度.

表 1 近似解与有限分析数值计算结果对比

Table 1 Numerical results comparison between approximate solution and FEA solution

边界条件	等效刚度 (N/m)			等效频率 (Hz)		
	有限元法	本方法	偏差	有限元法	本方法	偏差
两端简支	95903.77	97438.14	1.60	914.91	929.03	1.54
两端固支	382718.37	385099.76	0.62	2071.37	2112.05	1.96
固支简支	218958.14	220747.59	0.82	1428.52	1460.12	2.21

5 结论

刚度、频率与预应力梁所受张力之间的关系是非常重要的物理关系, 本文针对纯弯曲问题静态变形容易求取, 而纯张力问题频率容易求取的特点, 综合了瑞利黎兹法与直接刚度法的优点, 首先求得弯曲变形和等效弯曲刚度, 再将静态变形作为振型, 通过动能等效得到梁的等效质量系数. 进一步根据纯张紧问题的频率公式, 计算出给定变形条件下的张力所引起的刚度, 最后叠加弯曲刚度和张力引起刚度获得总的刚度和频率. 数值算例表明, 本方法相对误差仅为 2.21%, 计算方法简单有效. 本文获得了三种典型边界条件下预应力梁的刚度、频率与张力的关系, 为相关传感器的设计计算提供了参考. 本文方法的优势是计算简单, 公式中的常量都是通过积分计算得到, 不需要实验参数拟合, 为预应力作用下的刚度频率分析提供了一种简便有效的计算方法.

本文方法适应于能够获得变形函数的任何截面梁. 由于预应力梁主要应用于动态特性可调的柔性铰链执行器和用于力测量的共振微梁传感器中, 其工作振型均为第一阶模态, 因此本文中的等效质量和等效频率公式推导仅考虑一阶固有频率.

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Approximate analytical solution of equivalent stiffness and natural frequency for prestressed beams

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Prestressed beams are widely used in micro-sensors and actuators, the relationship between the stiffness, natural frequency and prestress is an important physical relationship. It is easy to obtain static deformation for pure bending, and vibration frequency for pure tension. In this paper, based on assumption that the tension force does not affect the bending deformation, and mode shape does not affect the frequency of tension, we proposed a new approximate solution, and find the solution for three different boundary conditions including clamped-clamped, hinged-hinged and clamped-hinged. Compared with the finite element analysis results, the relative error is less than 2.21%. The presented method is simplify but efficient for estimate the equivalent stiffness and natural frequency of prestressed beams.

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