论文

磁电弹材料反平面切口奇性数值分析

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摘要 磁电弹复合材料具有压电压磁的特点而被用于制作传感器等智能元件, 在这些元器件中经常遇到界面端或切口问题, 切口处由于易产生较高的奇异场而导致机械失效或电介击穿致使器件失效. 本文主要研究磁电弹材料反平面切口的奇性问题. 基于切口根部物理场的渐近展开假设, 从应力平衡方程和电磁麦克斯韦方程组出发, 导出了关于磁电弹材料反平面切口奇性指数的特征微分方程组, 并将切口的力电磁学边界条件以及粘接材料的界面协调条件表达为奇性指数和特征角函数的组合. 磁电弹材料切口反平面奇性指数的计算被转化为相应边界条件下常微分方程组特征值的求解问题, 通过插值矩阵法计算出各阶奇性指数和相应的特征角函数. 计算结果能提供减小切口奇性程度的切口角度和材料组合方案, 更好地指导磁电弹智能元器件的结构设计.

关键词 磁电弹材料, 反平面, 切口, 奇性分析

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磁电弹材料是一种具有磁场、电场与弹性场耦合性质的新型复合材料, 可用于制作传感器、驱动器或电磁存储元件等. 在制作和使用这些元器件的过程中会形成各种切口或异质材料界面端(如图 1(a)), 切口端部较高的奇异场常导致器件的机械失效或电介击穿^[1,2]. 近年来, 磁电弹材料元器件中界面端部开裂等由奇异场引起的破坏受到重点关注^[3].

裂纹作为切口张角为零度的特例首先被进行了详细的研究. Zhao 等人^[4]研究了三维双磁电弹材料中任意形状界面裂纹尖端的奇性指数和奇异行为. Fan 等人^[5]采用广义 Stroh 变换研究了二维磁电弹粘接材

料界面裂纹尖端力电磁场的奇异特征. 孙建亮等人^[6]分析了功能梯度压电/压磁材料中裂纹在反平面剪切载荷下的断裂问题. 胡克强等人^[7]研究了矩形压电体中反平面裂纹的电弹性场. 马鹏等人^[8]利用积分变换和奇异积分方程技术研究了压电压磁双材料界面裂纹二维断裂问题. Wan 等人^[9]将垂直磁电弹双材料界面的反平面裂纹问题转化为黎曼-希尔伯特问题, 并采用复变函数求解了裂尖力学、电学和磁学应力强度因子. Sánchez 等人^[10]采用边界元法研究了磁电弹复合材料的断裂问题. 平学成和陈梦成等人^[11]开创了一种杂交元分析了压电材料反平面界面裂纹问题.

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图 1 双磁电弹材料切口

(a) 元器件中常见切口类型; (b) 典型切口

Figure 1 Magneto-electro-elastic(MEE) bimaterial notch.

Guo 等人^[12]采用映射函数和复变函数法, 分析了磁电弹体中环形洞口萌生的多裂纹反平面问题. Li^[13]采用傅里叶变换和拉普拉斯变换, 进行了在反平面机械荷载和平面内电、磁冲击下磁电弹材料中裂纹的动态响应分析. Sih 等人^[14]研究了在钛酸钡-铁酸钴复合材料中电磁极化对裂纹扩展的影响.

对于各向同性和各向异性弹性材料切口^[15]、压电材料切口^[16]尖端的奇性问题也都已得到广泛研究. 但磁电弹材料切口由于更复杂的耦合目前研究较少, Sue 等人^[17]通过复势函数法和特征展开法分析了反平面磁电弹粘接切口的应力奇性指数, 发现与弹性各向异性粘接切口奇性指数有显著的不同. Liu 等人^[18]采用 Mellin 变换导出了奇性指数的特征方程, 但这些特征方程是一组超越方程, 获得其解析解也是非常困难的.

我们曾提出引入位移渐近展开假设, 将平衡方程转化为特征方程, 采用插值矩阵法求解获取双弹性材料切口奇性场的方法^[19]. 对于受反平面机械荷载、平面内电场和磁场作用的磁电弹材料切口尖端的奇性分析问题, 亦可基于切口根部物理场的渐近展开假设, 将力学、电学和磁学理论控制方程转换成常微分方程组特征值问题, 然后采用数值方法求解, 获得磁电弹切口的各阶奇性指数, 并可同时求出相应物理量的特征角函数.

1 切口奇性特征方程的建立

将主轴系下磁电弹材料反平面问题的物理关系^[18]在如图 1(b)所示的极坐标系 $or\theta$ 下进行变换, 则磁电弹材料反平面问题中切应力分量 σ_{rz} 、电位移分量 D_r 和磁感分量 B_r 与应变分量 γ_{rz} 、电场强度分量 E_r 和磁

场强度分量 $H_i(i=\theta, r)$ 之间的关系可表示为

$$\begin{bmatrix} \sigma_{rz} \\ \sigma_{\theta z} \\ D_r \\ D_\theta \\ B_r \\ B_\theta \end{bmatrix} = [Q(\theta)] \cdot \begin{bmatrix} r_{rz} \\ r_{\theta z} \\ E_r \\ E_\theta \\ H_r \\ H_\theta \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} & Q_{36} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} & Q_{45} & Q_{46} \\ Q_{51} & Q_{52} & Q_{53} & Q_{54} & Q_{55} & Q_{56} \\ Q_{61} & Q_{62} & Q_{63} & Q_{64} & Q_{65} & Q_{66} \end{bmatrix} \cdot \begin{bmatrix} r_{rz} \\ r_{\theta z} \\ E_r \\ E_\theta \\ H_r \\ H_\theta \end{bmatrix} \quad (1)$$

式(1)中矩阵 $[Q(\theta)]$ 的每一个元素是材料弹性常数 C_{44} 、压电常数 e_{15} 、压磁常数 q_{15} 、介电常数 ϵ_{11} 和磁导率 Γ_{11} 的组合, 且均为 θ 的函数. 式(1)中

$$\begin{aligned} r_{rz} &= \partial w / \partial r, \quad r_{\theta z} = \frac{1}{r} \frac{\partial w}{\partial \theta}, \quad E_r = -\partial \varphi / \partial r, \\ E_\theta &= -\frac{1}{r} \frac{\partial \varphi}{\partial \theta}, \quad H_r = -\partial \psi / \partial r, \quad H_\theta = -\frac{1}{r} \frac{\partial \psi}{\partial \theta}. \end{aligned} \quad (2)$$

其中 w 为位移, φ 为电势, ψ 为磁势. 将式(2)代入式(1)可以得到用位移、电势和磁势表达的应力、电位移和磁感分量

$$\begin{aligned} \sigma_{rz} &= Q_{11} \frac{\partial w}{\partial r} + Q_{12} \frac{1}{r} \frac{\partial w}{\partial \theta} - Q_{13} \frac{\partial \varphi}{\partial r} - Q_{14} \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\ &\quad - Q_{15} \frac{\partial \psi}{\partial r} - Q_{16} \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \\ \sigma_{\theta z} &= Q_{21} \frac{\partial w}{\partial r} + Q_{22} \frac{1}{r} \frac{\partial w}{\partial \theta} - Q_{23} \frac{\partial \varphi}{\partial r} - Q_{24} \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\ &\quad - Q_{25} \frac{\partial \psi}{\partial r} - Q_{26} \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \\ D_r &= Q_{31} \frac{\partial w}{\partial r} + Q_{32} \frac{1}{r} \frac{\partial w}{\partial \theta} - Q_{33} \frac{\partial \varphi}{\partial r} - Q_{34} \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\ &\quad - Q_{35} \frac{\partial \psi}{\partial r} - Q_{36} \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \\ D_\theta &= Q_{41} \frac{\partial w}{\partial r} + Q_{42} \frac{1}{r} \frac{\partial w}{\partial \theta} - Q_{43} \frac{\partial \varphi}{\partial r} - Q_{44} \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\ &\quad - Q_{45} \frac{\partial \psi}{\partial r} - Q_{46} \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \\ B_r &= Q_{51} \frac{\partial w}{\partial r} + Q_{52} \frac{1}{r} \frac{\partial w}{\partial \theta} - Q_{53} \frac{\partial \varphi}{\partial r} - Q_{54} \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\ &\quad - Q_{55} \frac{\partial \psi}{\partial r} - Q_{56} \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \end{aligned} \quad (3)$$

$$\begin{aligned}
& -Q_{55} \frac{\partial \psi}{\partial r} - Q_{56} \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \\
B_\theta &= Q_{61} \frac{\partial w}{\partial r} + Q_{62} \frac{1}{r} \frac{\partial w}{\partial \theta} - Q_{63} \frac{\partial \varphi}{\partial r} - Q_{64} \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \\
& - Q_{65} \frac{\partial \psi}{\partial r} - Q_{66} \frac{1}{r} \frac{\partial \psi}{\partial \theta}.
\end{aligned}$$

不考虑体力、自由电荷和电荷密度时, 磁电弹材料反平面问题的应力平衡方程和电磁麦克斯韦方程可以表示为

$$\begin{aligned}
\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\sigma_{rz}}{r} &= 0, \\
\frac{\partial D_r}{\partial r} + \frac{1}{r} \frac{\partial D_\theta}{\partial \theta} + \frac{D_r}{r} &= 0, \\
\frac{\partial B_r}{\partial r} + \frac{1}{r} \frac{\partial B_\theta}{\partial \theta} + \frac{B_r}{r} &= 0.
\end{aligned} \quad (4)$$

将式(3)代入式(4), 可以得到用位移、电势和磁势表达的微分方程组

$$\begin{aligned}
& Q_{11} \frac{\partial^2 w}{\partial r^2} + \frac{Q_{22}}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{(Q_{12} + Q_{21})}{r} \frac{\partial^2 w}{\partial r \partial \theta} \\
& + \frac{(Q_{11} + Q'_{21})}{r} \frac{\partial w}{\partial r} + \frac{Q'_{22}}{r^2} \frac{\partial w}{\partial \theta} - Q_{13} \frac{\partial^2 \varphi}{\partial r^2} \\
& - \frac{Q_{24}}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} - \frac{(Q_{14} + Q_{23})}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} \\
& - \frac{(Q_{13} + Q'_{23})}{r} \frac{\partial \varphi}{\partial r} - \frac{Q'_{24}}{r^2} \frac{\partial \varphi}{\partial \theta} - Q_{15} \frac{\partial^2 \psi}{\partial r^2} \\
& - \frac{Q_{26}}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{(Q_{16} + Q_{25})}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} \\
& - \frac{(Q_{15} + Q'_{25})}{r} \frac{\partial \psi}{\partial r} - \frac{Q'_{26}}{r^2} \frac{\partial \psi}{\partial \theta} = 0, \\
& Q_{31} \frac{\partial^2 w}{\partial r^2} + \frac{Q_{42}}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{(Q_{32} + Q_{41})}{r} \frac{\partial^2 w}{\partial r \partial \theta} \\
& + \frac{(Q_{31} + Q'_{41})}{r} \frac{\partial w}{\partial r} + \frac{Q'_{42}}{r^2} \frac{\partial w}{\partial \theta} - Q_{33} \frac{\partial^2 \varphi}{\partial r^2} \\
& - \frac{Q_{44}}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} - \frac{(Q_{34} + Q_{43})}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} \\
& - \frac{(Q_{33} + Q'_{43})}{r} \frac{\partial \varphi}{\partial r} - \frac{Q'_{44}}{r^2} \frac{\partial \varphi}{\partial \theta} - Q_{35} \frac{\partial^2 \psi}{\partial r^2} \\
& - \frac{Q_{46}}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{(Q_{36} + Q_{45})}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} \\
& - \frac{(Q_{35} + Q'_{45})}{r} \frac{\partial \psi}{\partial r} - \frac{Q'_{46}}{r^2} \frac{\partial \psi}{\partial \theta} = 0, \\
& Q_{51} \frac{\partial^2 w}{\partial r^2} + \frac{Q_{62}}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{(Q_{52} + Q_{61})}{r} \frac{\partial^2 w}{\partial r \partial \theta}
\end{aligned} \quad (5)$$

$$\begin{aligned}
& + \frac{(Q_{51} + Q'_{61})}{r} \frac{\partial w}{\partial r} + \frac{Q'_{62}}{r^2} \frac{\partial w}{\partial \theta} - Q_{53} \frac{\partial^2 \varphi}{\partial r^2} \\
& - \frac{Q_{64}}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} - \frac{(Q_{54} + Q_{63})}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} \\
& - \frac{(Q_{53} + Q'_{63})}{r} \frac{\partial \varphi}{\partial r} - \frac{Q'_{64}}{r^2} \frac{\partial \varphi}{\partial \theta} \\
& - Q_{55} \frac{\partial^2 \psi}{\partial r^2} - \frac{Q_{66}}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \frac{(Q_{56} + Q_{65})}{r} \frac{\partial^2 \psi}{\partial r \partial \theta} \\
& - \frac{(Q_{55} + Q'_{65})}{r} \frac{\partial \psi}{\partial r} - \frac{Q'_{66}}{r^2} \frac{\partial \psi}{\partial \theta} = 0.
\end{aligned}$$

式中 $(\cdots)'$ 表示对坐标 θ 的一阶导。

将切口尖端区域的位移场、电势场和磁势场表达成如下的级数渐近展开形式^[17]

$$\begin{aligned}
w &= \sum_{k=1}^N A_k r^{\lambda_k} \tilde{w}_k(\lambda_k, \theta), \\
\varphi &= \sum_{k=1}^N A_k r^{\lambda_k} \tilde{\varphi}_k(\lambda_k, \theta), \\
\psi &= \sum_{k=1}^N A_k r^{\lambda_k} \tilde{\psi}_k(\lambda_k, \theta).
\end{aligned} \quad (6)$$

其中 r 为距切口尖端的径向距离, A_k 为组合幅值系数, λ_k 为切口尖端的奇性指数, N 表示截取的级数项数, $\tilde{w}_k(\lambda_k, \theta)$, $\tilde{\varphi}_k(\lambda_k, \theta)$ 和 $\tilde{\psi}_k(\lambda_k, \theta)$ 分别为位移、电势和磁势的特征角函数。取式(6)中的典型项 $A_k r^{\lambda_k} \tilde{w}_k(\lambda_k, \theta)$, $A_k r^{\lambda_k} \tilde{\varphi}_k(\lambda_k, \theta)$ 和 $A_k r^{\lambda_k} \tilde{\psi}_k(\lambda_k, \theta)$ 代入式(5), 化简得(为书写简洁, 以下将 $\tilde{w}_k(\lambda_k, \theta)$, $\tilde{\varphi}_k(\lambda_k, \theta)$ 、 $\tilde{\psi}_k(\lambda_k, \theta)$ 和 λ_k 分别简写为 $\tilde{w}$, $\tilde{\varphi}$, $\tilde{\psi}$ 和 λ)

$$\begin{aligned}
& Q_{22} \tilde{w}'' - Q_{24} \tilde{\varphi}'' - Q_{26} \tilde{\psi}'' + Q'_{22} \tilde{w}' - Q'_{24} \tilde{\varphi}' - Q'_{26} \tilde{\psi}' \\
& + \lambda(Q_{12} + Q_{21}) \tilde{w}' - \lambda(Q_{14} + Q_{23}) \tilde{\varphi}' - \lambda(Q_{16} + Q_{25}) \tilde{\psi}' \\
& + \lambda^2 Q_{11} \tilde{w} - \lambda^2 Q_{13} \tilde{\varphi} - \lambda^2 Q_{15} \tilde{\psi} + \lambda Q'_{21} \tilde{w} - \lambda Q'_{23} \tilde{\varphi} \\
& - \lambda Q'_{25} \tilde{\psi} = 0, \\
& Q_{42} \tilde{w}'' - Q_{44} \tilde{\varphi}'' - Q_{46} \tilde{\psi}'' + Q'_{42} \tilde{w}' - Q'_{44} \tilde{\varphi}' - Q'_{46} \tilde{\psi}' \\
& + \lambda(Q_{32} + Q_{41}) \tilde{w}' - \lambda(Q_{34} + Q_{43}) \tilde{\varphi}' - \lambda(Q_{36} + Q_{45}) \tilde{\psi}' \\
& + \lambda^2 Q_{31} \tilde{w} - \lambda^2 Q_{33} \tilde{\varphi} - \lambda^2 Q_{35} \tilde{\psi} + \lambda Q'_{41} \tilde{w} - \lambda Q'_{43} \tilde{\varphi} \\
& - \lambda Q'_{45} \tilde{\psi} = 0, \\
& Q_{62} \tilde{w}'' - Q_{64} \tilde{\varphi}'' - Q_{66} \tilde{\psi}'' + Q'_{62} \tilde{w}' - Q'_{64} \tilde{\varphi}' - Q'_{66} \tilde{\psi}' \\
& + \lambda(Q_{52} + Q_{61}) \tilde{w}' - \lambda(Q_{54} + Q_{63}) \tilde{\varphi}' - \lambda(Q_{56} + Q_{65}) \tilde{\psi}' \\
& + \lambda^2 Q_{51} \tilde{w} - \lambda^2 Q_{53} \tilde{\varphi} - \lambda^2 Q_{55} \tilde{\psi} + \lambda Q'_{61} \tilde{w} - \lambda Q'_{63} \tilde{\varphi} \\
& - \lambda Q'_{65} \tilde{\psi} = 0.
\end{aligned} \quad (7)$$

式中 $(\cdots)''$ 表示对坐标 θ 的二阶导。从而, 磁电弹材料切口奇性指数的计算变成了求解常微分方程组式(7)

的特征值问题.

2 边界条件和界面连续条件

2.1 A 类边界条件

该类边界条件假设边界上应力、电位移和磁导均自由, 即

$$\sigma_{\theta z} = D_{\theta} = B_{\theta} = 0. \quad (8)$$

注意到式(3), 并取式(6)中 w, ϕ, ψ 的渐近展开式中的典型项代入上式, 化简得

$$\begin{aligned} Q_{22}\tilde{w}' - Q_{24}\tilde{\phi}' - Q_{26}\tilde{\psi}' + \lambda Q_{21}\tilde{w} - \lambda Q_{23}\tilde{\phi} - \lambda Q_{25}\tilde{\psi} &= 0, \\ Q_{42}\tilde{w}' - Q_{44}\tilde{\phi}' - Q_{46}\tilde{\psi}' + \lambda Q_{41}\tilde{w} - \lambda Q_{43}\tilde{\phi} - \lambda Q_{45}\tilde{\psi} &= 0, \quad (9) \\ Q_{62}\tilde{w}' - Q_{64}\tilde{\phi}' - Q_{66}\tilde{\psi}' + \lambda Q_{61}\tilde{w} - \lambda Q_{63}\tilde{\phi} - \lambda Q_{65}\tilde{\psi} &= 0. \end{aligned}$$

2.2 B 类边界条件

此类边界条件假设边界位移约束、电位移和磁导自由, 即

$$w = D_{\theta} = B_{\theta} = 0. \quad (10)$$

注意到式(3), 并取式(6)中 w, ϕ, ψ 的渐近展开式中的典型项代入上式, 化简得

$$\begin{aligned} \tilde{w} &= 0, \\ Q_{42}\tilde{w}' - Q_{44}\tilde{\phi}' - Q_{46}\tilde{\psi}' + \lambda Q_{41}\tilde{w} - \lambda Q_{43}\tilde{\phi} - \lambda Q_{45}\tilde{\psi} &= 0, \quad (11) \\ Q_{62}\tilde{w}' - Q_{64}\tilde{\phi}' - Q_{66}\tilde{\psi}' + \lambda Q_{61}\tilde{w} - \lambda Q_{63}\tilde{\phi} - \lambda Q_{65}\tilde{\psi} &= 0. \end{aligned}$$

2.3 界面连续条件

如图 1(b)所示的双磁电弹材料粘结切口, 假设界面为理想粘结, 界面上的连续条件可以表示为(以下公式中出现的(1), (2)均表示属于第 1 种、第 2 种材料)

$$\begin{aligned} w_{(1)} &= w_{(2)}, \quad \phi_{(1)} = \phi_{(2)}, \quad \psi_{(1)} = \psi_{(2)}, \\ \sigma_{\theta z(1)} &= \sigma_{\theta z(2)}, \quad D_{\theta(1)} = D_{\theta(2)}, \quad B_{\theta(1)} = B_{\theta(2)}. \end{aligned} \quad (12)$$

注意到式(3), 并将切口尖端物理场渐近展开式(6)中的典型项代入式(12), 可将界面连续条件转变为

$$\begin{aligned} \tilde{w}_{(1)} &= \tilde{w}_{(2)}, \\ \tilde{\phi}_{(1)} &= \tilde{\phi}_{(2)}, \\ \tilde{\psi}_{(1)} &= \tilde{\psi}_{(2)}, \\ Q_{22}^{(1)}\tilde{w}_{(1)}' - Q_{24}^{(1)}\tilde{\phi}_{(1)}' - Q_{26}^{(1)}\tilde{\psi}_{(1)}' - Q_{22}^{(2)}\tilde{w}_{(2)}' + Q_{24}^{(2)}\tilde{\phi}_{(2)}' \\ &+ Q_{26}^{(2)}\tilde{\psi}_{(2)}' + \lambda Q_{21}^{(1)}\tilde{w}_{(1)} - \lambda Q_{23}^{(1)}\tilde{\phi}_{(1)} - \lambda Q_{25}^{(1)}\tilde{\psi}_{(1)} \\ &- \lambda Q_{21}^{(2)}\tilde{w}_{(2)} + \lambda Q_{23}^{(2)}\tilde{\phi}_{(2)} + \lambda Q_{25}^{(2)}\tilde{\psi}_{(2)} = 0, \quad (13) \\ Q_{42}^{(1)}\tilde{w}_{(1)}' - Q_{44}^{(1)}\tilde{\phi}_{(1)}' - Q_{46}^{(1)}\tilde{\psi}_{(1)}' - Q_{42}^{(2)}\tilde{w}_{(2)}' + Q_{44}^{(2)}\tilde{\phi}_{(2)}' \\ &- Q_{46}^{(2)}\tilde{\psi}_{(2)}' + \lambda Q_{41}^{(1)}\tilde{w}_{(1)} - \lambda Q_{43}^{(1)}\tilde{\phi}_{(1)} - \lambda Q_{45}^{(1)}\tilde{\psi}_{(1)} \\ &- \lambda Q_{41}^{(2)}\tilde{w}_{(2)} + \lambda Q_{43}^{(2)}\tilde{\phi}_{(2)} + \lambda Q_{45}^{(2)}\tilde{\psi}_{(2)} = 0. \end{aligned}$$

$$\begin{aligned} &+ Q_{46}^{(2)}\tilde{\psi}_{(2)}' + \lambda Q_{41}^{(1)}\tilde{w}_{(1)} - \lambda Q_{43}^{(1)}\tilde{\phi}_{(1)} - \lambda Q_{45}^{(1)}\tilde{\psi}_{(1)} \\ &- \lambda Q_{41}^{(2)}\tilde{w}_{(2)} + \lambda Q_{43}^{(2)}\tilde{\phi}_{(2)} + \lambda Q_{45}^{(2)}\tilde{\psi}_{(2)} = 0, \\ Q_{62}^{(1)}\tilde{w}_{(1)}' - Q_{64}^{(1)}\tilde{\phi}_{(1)}' - Q_{66}^{(1)}\tilde{\psi}_{(1)}' - Q_{62}^{(2)}\tilde{w}_{(2)}' + Q_{64}^{(2)}\tilde{\phi}_{(2)}' \\ &+ Q_{66}^{(2)}\tilde{\psi}_{(2)}' + \lambda Q_{61}^{(1)}\tilde{w}_{(1)} - \lambda Q_{63}^{(1)}\tilde{\phi}_{(1)} - \lambda Q_{65}^{(1)}\tilde{\psi}_{(1)} \\ &- \lambda Q_{61}^{(2)}\tilde{w}_{(2)} + \lambda Q_{63}^{(2)}\tilde{\phi}_{(2)} + \lambda Q_{65}^{(2)}\tilde{\psi}_{(2)} = 0. \end{aligned}$$

3 切口特征微分方程组的求解

由于磁电弹材料切口奇性指数特征方程组式(7)中出现了 λ^2 项, 这是一个非线性特征值问题. 特引入以下三个变量

$$\begin{aligned} \tilde{g}_w &= \lambda \tilde{w}, \\ \tilde{g}_{\phi} &= \lambda \tilde{\phi}, \\ \tilde{g}_{\psi} &= \lambda \tilde{\psi}. \end{aligned}$$

将式(7)的非线性特征常微分方程组转化为线性问题

$$\begin{aligned} &Q_{22}\tilde{w}'' - Q_{24}\tilde{\phi}'' - Q_{26}\tilde{\psi}'' + Q_{22}'\tilde{w}' - Q_{24}'\tilde{\phi}' - Q_{26}'\tilde{\psi}' \\ &+ \lambda(Q_{12} + Q_{21})\tilde{w}' - \lambda(Q_{14} + Q_{23})\tilde{\phi}' - \lambda(Q_{16} + Q_{25})\tilde{\psi}' \\ &+ \lambda Q_{11}\tilde{g}_w - \lambda Q_{13}\tilde{g}_{\phi} - \lambda Q_{15}\tilde{g}_{\psi} + \lambda Q_{21}'\tilde{w} - \lambda Q_{23}'\tilde{\phi} \\ &- \lambda Q_{25}'\tilde{\psi} = 0, \\ &Q_{42}\tilde{w}'' - Q_{44}\tilde{\phi}'' - Q_{46}\tilde{\psi}'' + Q_{42}'\tilde{w}' - Q_{44}'\tilde{\phi}' - Q_{46}'\tilde{\psi}' \\ &+ \lambda(Q_{32} + Q_{41})\tilde{w}' - \lambda(Q_{34} + Q_{43})\tilde{\phi}' - \lambda(Q_{36} + Q_{45})\tilde{\psi}' \\ &+ \lambda Q_{31}\tilde{g}_w - \lambda Q_{33}\tilde{g}_{\phi} - \lambda Q_{35}\tilde{g}_{\psi} + \lambda Q_{41}'\tilde{w} - \lambda Q_{43}'\tilde{\phi} \\ &- \lambda Q_{45}'\tilde{\psi} = 0, \\ &Q_{62}\tilde{w}'' - Q_{64}\tilde{\phi}'' - Q_{66}\tilde{\psi}'' + Q_{62}'\tilde{w}' - Q_{64}'\tilde{\phi}' - Q_{66}'\tilde{\psi}' \\ &+ \lambda(Q_{52} + Q_{61})\tilde{w}' - \lambda(Q_{54} + Q_{63})\tilde{\phi}' - \lambda(Q_{56} + Q_{65})\tilde{\psi}' \\ &+ \lambda Q_{51}\tilde{g}_w - \lambda Q_{53}\tilde{g}_{\phi} - \lambda Q_{55}\tilde{g}_{\psi} + \lambda Q_{61}'\tilde{w} - \lambda Q_{63}'\tilde{\phi} \\ &- \lambda Q_{65}'\tilde{\psi} = 0. \end{aligned} \quad (14)$$

文献[19]开发了一种称为插值矩阵法的数值方法来求解常微分方程组特征值问题. 该法将求解区间剖分为若干个子区间, 在每个子区间上使用低阶分段多项式函数插值逼近待求函数, 以常微分方程组中出现的最高阶导数在离散节点上的值作为离散系统的未知参数, 形成代数特征值方程组, 解之可以获得特征值与相应的特征函数. 插值矩阵法可以同时求解出常微分方程组特征值问题中出现的所有函数和其各阶导数, 且具有同阶精度. 这里采用插值矩阵法来求解在相应的边界条件(式(9)或(11))、界面连续条件(式(13))下, 磁电弹切口奇性指数的特征常微分方程组(式(14)).

4 数值算例

选择以压磁材料 CoFe_2O_4 为基体、压电材料 BaTiO_3 为夹杂构成的磁电弹性复合材料, 其等效材料常数可以采用场平均方法^[18]或细观力学方法^[20]获得, 为数值计算方便, 这里采用场平均的方法, 磁电弹性复合材料的等效材料常数根据 CoFe_2O_4 和 BaTiO_3 的体积百分比按下式合成^[18]

$$K_{ij}^C = K_{ij}^I V_f + K_{ij}^M (1 - V_f). \quad (15)$$

其中, V_f 为夹杂 BaTiO_3 的体积分数, K_{ij}^C , K_{ij}^I 和 K_{ij}^M 分别代表磁电弹复合材料、夹杂和基体的材料常数 C_{ij} , e_{ij} , q_{ij} , ε_{ij} , λ_{ij} 和 Γ_{ij} , 具体数值见表 1.

表 1 BaTiO_3 和 CoFe_2O_4 的材料常数

Table 1 Material constants of BaTiO_3 and CoFe_2O_4

Material constant	BaTiO_3	CoFe_2O_4
C_{44} (N/m ²)	4.3×10^{10}	4.53×10^{10}
e_{15} (C/m ²)	1.16×10	0.00
q_{15} (N/Am)	0.00	5.50×10^2
ε_{11} (C/Vm)	1.12×10^{-8}	0.80×10^{-10}
Γ_{11} (NS ² /C ²)	5.00×10^{-6}	1.00×10^{-4}

表 2 不同子区间划分时单磁电弹材料切口奇性指数计算结果(A-A 边界条件)

Table 2 The calculated singularity orders by different sub-interval numbers for single-material MEE notch under the A-A boundary condition

$\alpha (= \beta)$	Methods	λ_1	λ_4	λ_7
100°	IMM, $n=10$	-0.0999	0.8045	1.7334
	IMM, $n=20$	-0.1000	0.8003	1.7021
	IMM, $n=40$	-0.1000	0.8000	1.7001
	IMM, $n=50$	-0.1000	0.8000	1.7001
	ref.[18]	-0.1000	—	—
120°	IMM, $n=10$	-0.2499	0.5037	1.2778
	IMM, $n=20$	-0.2500	0.5002	1.2517
	IMM, $n=40$	-0.2500	0.5000	1.2500
	IMM, $n=50$	-0.2500	0.5000	1.2500
	ref.[18]	-0.2500	—	—
140°	IMM, $n=10$	-0.3570	0.2889	0.9524
	IMM, $n=20$	-0.3571	0.2859	0.9301
	IMM, $n=40$	-0.3571	0.2857	0.9286
	IMM, $n=50$	-0.3571	0.2857	0.9286
	ref.[18]	-0.3571	—	—
160°	IMM, $n=10$	-0.4374	0.1278	0.7084
	IMM, $n=20$	-0.4375	0.1252	0.6888
	IMM, $n=40$	-0.4375	0.1250	0.6875
	IMM, $n=50$	-0.4375	0.1250	0.6875
	ref.[18]	-0.4375	—	—
180°	IMM, $n=10$	-0.4999	0.0025	0.5185
	IMM, $n=20$	-0.5000	0.0002	0.5012
	IMM, $n=40$	-0.5000	0.0000	0.5000
	IMM, $n=50$	-0.5000	0.0000	0.5000
	ref.[18]	-0.5000	—	—

例1 单磁电弹材料切口.

本例取体积分数 $V_f = 10\%$, 根据式(15)计算出磁电弹材料的各等效常数, 分别研究 A-A, B-B 和 A-B 三种不同边界条件下切口的奇异性问题. 本文方法在采用插值矩阵法^[19](IMM)求解关于切口奇性指数的特征微分方程组时, 需要将切口的弧线边界(见图 1(b))划分为若干(n)个子区间. 表 2 列出了在 A-A 边界条件下不同子区间划分时切口奇性指数前 9 阶的计算结果, 由于每一个奇性指数都有两个对应的重根, 所以表中仅列出了第 1, 4, 7 阶奇性指数, 计算发现, 前 3 阶为奇性指数, 后面的高阶为非奇性指数, 这些高阶非奇性指数的获取, 为计算切口尖端的完整磁电弹场奠定了基础. 文献[18]导出了 $\alpha = \beta$ 时切口奇性指数的理论解, 但那是一个高阶超越方程, 求解特别困难, 该文仅提供了第 1 阶奇性指数, 现同列于表 2 作为参考. 从表 2 中可以看出, 对于低阶奇性指数, 插值矩阵法将弧线边界剖分 20 段, 在所选的有效数字范围内计算结果和参考解一致, 高阶非奇性指数没有参考解可以对照, 但切口弧线边界剖分

40 段和 50 段的计算结果收敛, 可以认为切口弧线边界剖分 40 段时本文方法即可获得足够高的计算精度, 因此, 以下计算均按插值矩阵法在切口弧线边界上剖分 40 段进行.

图 2(a)列出了在 B-B 边界条件下单磁电弹材料切口前 9 阶奇性指数的计算值, 和 A-A 边界条件下切口的奇性指数分布相同, 每一个奇性指数都有两个对应的重根, 所以图 2(a)也仅给出了第 1, 4, 7 阶奇性指数. 计算发现切口张角(指 $\alpha + \beta$ 的值)大于 180° 时才开始出现奇异性, 这和弹性材料切口奇异状况相同. 图 2(b)绘出了在 A-B 边界条件下单磁电弹材料切口前 4 阶奇性指数, 其中第 2, 3 阶奇性指数相同. 在 A-B 边界条件下, 切口张角大于 90° 就出现奇异性,

这与弹性材料以及压电材料切口显著不同, 切口张角大于 180° 后将出现两个奇性指数、张角大于 260° 后将出现三个奇性指数. 将文献[18]给出的第 1 阶奇性指数同列于图 2, 比较发现本文结果能和它保持一致.

在计算出切口奇性指数的同时, 本文方法还可以获得对应的奇性特征角函数, 而这是文献[18]所不能胜任的. 图 3 给出了在 A-A 边界条件下, 单磁电弹材料裂纹($\alpha = \beta = 180^\circ$)第 1 阶奇性指数对应的特征角函数及其一阶导函数的数值计算结果. 至此, 切口尖端物理场的级数渐近展开式(6)中, 奇性指数和特征角函数全部获得, 仅组合幅值系数未知. 配合在切口远场开展的边界元法等数值方法分析, 可以计算出该组合系数, 从而可获得切口尖端的完整奇异物理场.

图 2 (网络版彩图)单磁电弹材料切口奇性指数随张角的变化

(a) B-B 边界条件; (b) A-B 边界条件

Figure 2 (Color online) The variation of singularity orders for a single-material MEE notch versus notch angle.

图 3 (网络版彩图)A-A 边界条件下磁电弹材料裂纹第 1 阶奇性指数对应的特征角函数

(a) 特征角函数; (b) 特征角函数的一阶导函数

Figure 3 (Color online) The eigen angular functions corresponding to the first singularity order for MEE crack under the A-A boundary condition.

例 2 双磁电弹材料切口.

对于双磁电弹材料切口, 在每种材料所在区域内可各列一组常微分特征方程组式(14), 利用界面连续条件式(13)将这两组方程组联合起来, 并配合相应的边界条件式(9)或式(11), 利用插值矩阵法求解就可以计算出双磁电弹材料切口的奇性指数.

本例通过改变夹杂 BaTiO_3 的体积分数以获取不同的磁电弹复合材料, 第 1, 2 两种材料中夹杂 BaTiO_3 的体积分数分别记为 $V_f^{(1)}$ 和 $V_f^{(2)}$, 将两种不同的复合材料粘结以形成双磁电弹材料切口. 图 4 绘出了在 A-A 边界条件下, $(V_f^{(1)}, V_f^{(2)})$ 分别等于 (20%, 50%), (90%, 10%) 时这两种不同双材料切口奇性指数

随切口张角的变化, 其中令 $\alpha=180^\circ$ 不变, 仅改变 β 值的大小. 文献[18]给出了关于双磁电弹材料切口奇性指数的隐式方程, 但没有提供该超越方程的解法, 这里采用牛顿迭代法来计算该式获取结果与之对照. 对 $(V_f^{(1)}, V_f^{(2)})$ 等于 (20%, 50%) 的组合, 在 $(-1, 0)$ 之间存在三个奇性指数, 其中前两个为重根, 对于 (90%, 10%) 组合, 在 $(-1, 0)$ 之间也存在三个奇性指数, 但后两个为重根. 从图 4 可以看出本文方法具有足够高的计算精度.

图 5 给出了体积分数对切口最强阶奇性指数的影响, 分别计算 $\alpha=90^\circ, \beta=180^\circ$ 和 $\alpha=90^\circ, \beta=270^\circ$ 两种不同角度下双材料切口的奇性指数. 结果发现, 随着

图 4 (网络版彩图)切口开角对奇性指数的影响($\alpha=180^\circ$)

(a) $V_f^{(2)} = 20\% / V_f^{(1)} = 50\%$; (b) $V_f^{(1)} = 90\% / V_f^{(2)} = 10\%$

Figure 4 (Color online) The variation of singularity orders of a bi-material MEE notch versus notch angle when $\alpha = 180^\circ$

图 5 (网络版彩图)体积分数对切口最强阶奇性指数的影响

(a) $\alpha = 90^\circ, \beta = 180^\circ$; (b) $\alpha = 90^\circ, \beta = 270^\circ$

Figure 5 (Color online) The influence of the volume fraction on the strongest singularity order.

两种材料体积分数差别越大, 切口的奇异性越强. 注意到 $\alpha=90^\circ$, $\beta=270^\circ$ 时切口退化为裂纹问题, 当 $V_f^{(1)} = V_f^{(2)}$ 时进一步退化为单材料裂纹问题, 此时的最强阶奇性指数为 -0.5 , 和单弹性材料、单压电材料裂纹的奇性指数相同; 当 $V_f^{(1)} \neq V_f^{(2)}$ 时, 变成了异质材料裂纹问题, 计算发现异质程度越高奇异性越强, 当 $V_f^{(1)} = 0.9/V_f^{(2)} = 0.1$ 时裂纹的奇性指数为 -0.6290 .

5 结论

根据切口尖端位移、电势和磁势的渐近展开假设,

将磁电弹切口理论控制方程转变成一类常微分方程组特征值问题, 采用插值矩阵法求解, 可一次性地求出切口的各阶奇性指数、高阶非奇性指数和相应的特征角函数. 本文方法可以用于单、双材料切口奇性特征分析, 还可以退化分析裂纹尖端奇异性问题. 计算结果能提供减小奇性的切口角度和材料组合方案, 更好地指导磁电弹智能元器件的结构设计. 基于本文计算结果, 再配合边界元法等其他数值方法, 可计算出切口尖端附近包括非奇异项在内的完整应力场和完整电磁场, 为评估磁电弹元件结构的安全性提供基础.

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Numerical analysis of the singularities for a magneto-electro-elastic notch under anti-plane deformation

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Due to the capability of the conversion between mechanical, electric and magnetic energy, magneto-electro-elastic (MEE) materials can be applied to design sensors, actuators and other smart products. The notch and interfacial edge problem can always be encountered in these products. The singularity phenomenon is that the stress, electric displacement and magnetic induction may become theoretical infinite due to the materials or geometrical discontinuities at the vertex of notch. In practices, the mechanical failure or dielectric breakdown may initiate from the notch apex due to this high singularity. The present paper is trying to establish a new method to study the singularity at the vertex of notch encountered in MEE structures. Basing on the asymptotic assumption for the physical field near the notch tip, the singularities for a MEE notch under anti-plane deformation are investigated by the eigenfunction expansion approach. The characteristic differential equations respects to the singularities of the MEE notch are built from the equilibrium equations and Maxell equations. At the same time, the mechanical and electric boundary conditions together with the interfacial continuous conditions are expressed by the combination of the singularity orders and characteristic angle functions. Thus, the evaluation of the notch singularity orders is transformed into the problem of solving the ordinary differential equations under the designated boundary conditions. The singularity orders and the corresponding characteristic angle functions can be achieved by applying the interpolate matrix method to solve the obtained characteristic differential equations. The achieved results can provide the guidelines for the selection of notch angles and materials to reduce the singularity for the structure design of MEE products.

magneto-electro-elastic material, anti-plane, notch, singularity analysis

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