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p-th moment exponential stability of stochastic differential equations with impulse effect

SHEN LiJuan[1,2] & SUN JiTao[1]*

[1]*Department of Mathematics, Tongji University, Shanghai* 200092*, China;*
[2]*Department of Mathematics, Luoyang Normal University, Luoyang* 471022*, China*

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Abstract The p-th moment exponential stability of stochastic differential equations with impulse effect is addressed. By employing the method of vector Lyapunov functions, some sufficient conditions for the p-th moment exponential stability are established. In addition, the usual restriction of the growth rate of Lyapunov function is replaced by the condition of the drift and diffusion coefficients to study the p-th moment exponential stability. Several examples are also discussed to illustrate the effectiremess of the results obtained.

Keywords stochastic differential equations, impulse, p-th moment exponential stability, vector Lyapunov functions

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1 Introduction

Stochastic differential equations (SDEs) are well-known models to a wide variety of applications such as option pricing, forecast of the growth of population, etc. (see [1–3]). Stability of SDEs has been investigated by [4–6]. Ref. [7] gave a brief discussion on the practical stability and controllability of stochastic Markovian systems. The exponential mean-square stabilization of hybrid stochastic differential is studied in[8]. Recently, ref. [9] employed the Razumikhin techniques to investigate the stability of neutral stochastic functional differential equations.

Impulse effects exist widely in many evolution processes in which states are changing abruptly at certain moments of time, involving such fields as medicine, biology and electronics, etc. (see [10–12] and the references therein). However, quite often the state of systems are subject not only to sudden impulsive effect but also to stochastic perturbations, leading to stochastic differential equations with impulse effect. In the recent years, there has been an increasing interest in the investigation of SDEs with impulse effect, e.g., see [13, 14] and the references cited therein.

The stability theory of SDEs with impulse is rather complex mainly because of the stability notions derived from the random effect. The most important stability notions are mean-square stability, moment stability and almost sure stability. Moment stability, also called p-th moment stability, requires the convergence to zero with a moment of order p. When $p = 2$, the exponential stability is termed exponential

*Corresponding author (email: sunjt@sh163.net)

stability in mean square. Ref. [15] investigated p-th moment stability of stochastic differential equations with impulsive jump and Markovian switching. Xu et al. [16] studied p-th moment stability of stochastic differential equations with impulsive jump. Ref.[17] used Lyapunov-like functions and comparison theorem to study the global existence, uniqueness and stability.

However, to the best of our knowledge, few authors have considered the problem of p-th moment exponential stability of SDEs with impulse effect. Therefore, techniques and methods for p-th moment exponential stability should be developed.

As is well known, Lyapunov function can be used to get the property of the solution without solving the equation. Furthermore, vector Lyapunov functions offer more flexible mechanisms than a single function, since each component function can satisfy a less rigid requirement.

Motivated by the discussions above, in this paper, we will consider the SDEs with impulse effect and discuss the p-th moment exponential stability of the trivial solution by vector Lyapunov functions. Some sufficient conditions coupled with the frequency of impulsive effect are presented. Furthermore we'll try to study moment exponential stability by imposing conditions on the right-hand side functions instead of the condition on the growth rate of Lyapunov function. That will be an interesting study.

The rest of this paper is as follows. In section 2, we recall briefly some basic notations and preliminary facts. Section 3 is devoted to some sufficient conditions of p-th moment exponential stability. In section 4 some numerical examples are provided to illustrate the main result.

2 Preliminaries

In this section we will introduce some notations and definitions which will be used throughout the paper.

Here for a vector y, we denote by $|\cdot|$ the Euclidean norm. For a matrix M, its trace norm is denoted by $|M| = \sqrt{M^{\mathrm{T}}M}$. Assume $B(t)$ is an m-dimensional Brownian motion. $E = E_P$ denotes expectation with respect to the probability measure P.

Consider the stochastic differential equations with impulse effect as follows:

$$\begin{aligned} &\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}B(t), t \neq \tau_k,\\ &\Delta x(t) = I_k(x(t^-)), t = \tau_k, k \in \mathbb{N},\\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $t \in [t_0, \infty), t_0 < \tau_1 < \tau_2 < \ldots < \tau_k < \ldots$, with $\tau_k \to \infty$, $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ with $f(t, 0) = 0$ for all $t \geqslant t_0$, $g : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ with $g(t, 0) = 0$ for all $t \geqslant t_0$. $\Delta x(t)$ denotes the jump of x at t, i.e.

$$\Delta x(t) = x(t^+) - x(t^-) = x(t) - x(t^-).$$

$I_k : \mathbb{R}^n \to \mathbb{R}^n$ with $I_k(0) = 0$. The initial value x_0 is a random variable independent of the $B(s), s > 0$.

The following hypothesis is always assumed to hold throughout the paper.

H_1 There exists a path-wise unique right continuous stochastic process $x(t)$ with left limit satisfying (1) and $E|x(t, t_0, x_0)|^p < \infty, p > 0$ for all $t \geqslant t_0$.

Motivated by the previous work in [18] we give the following definition for (1).

Definition 1. For some $p > 0$, (1) is said to be p-th moment exponentially stable, if for every $\varepsilon > 0$, there exists $a > 0$ and $\delta = \delta(\varepsilon) > 0$ such that

$$E|x(t, t_0, x_0)|^p \leqslant \varepsilon \mathrm{e}^{-a(t-t_0)} \text{ for all } t \geqslant t_0,$$

whenever $E|x_0|^p < \delta$.

Unlike the terminology in [2], the terminology p-th moment exponential stability in Definition 1 is local.

To study the stability of (1), let us introduce some important notations. Let $C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^l_+)$ denote the family of all nonnegative functions $v(t, x)$ on $\mathbb{R} \times \mathbb{R}^n$, which are continuously twice differentiable in

x and once differentiable in t. For each $v(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^l_+)$, define an (infinitesimal) operator $Lv^i, i = 1, 2, \ldots, l$, along with (1) for the component of $v(t,x)$ as

$$Lv^i(t,x) = v^i_t(t,x) + v^i_x f(t,x) + \frac{1}{2} trace[g^T(t,x) v^i_{xx}(t,x) g(t,x)],$$

where

$$v^i_t(t,x) = \frac{\partial v^i(t,x)}{\partial t}, \quad v^i_{xx} = \left(\frac{\partial^2 v^i(t,x)}{\partial x_j \partial x_k}\right)_{n\times n},$$

and

$$v^i_x(t,x) = \Big(\frac{\partial v^i(t,x)}{\partial x_1}, \quad \frac{\partial v^i(t,x)}{\partial x_2}, \ldots, \frac{\partial v^i(t,x)}{\partial x_n}\Big).$$

Generally, 2-th exponential stability is called exponential stability in mean square.

3 Main results

Before giving the main results, we need the following lemma.

Lemma 1. Assume integrable function $\beta(t) : [t_0, \infty) \to \mathbb{R}$. For (1), if there exists $v(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^l_+)$ such that

$$\sum_{i=1}^{l} Lv^i(t,x) \leqslant \beta(t) \sum_{i=1}^{l} v^i(t,x),$$

for $t \in [\tau_{k-1}, \tau_k), k \in \mathbb{N}$, then for $t \in [\tau_{k-1}, \tau_k)$, we have

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant \sum_{i=1}^{l} Ev^i(\tau_{k-1}, x(\tau_{k-1})) \exp(\int_{\tau_{k-1}}^{t} \beta(s)\mathrm{d}s).$$

Proof. By the generalized Itô's formula we can easily prove it. Given small enough $h > 0$ such that $t, t+h \in [\tau_{k-1}, \tau_k)$, by the generalized Itô's formula, we have

$$\begin{aligned}\sum_{i=1}^{l} Ev^i(t+h, x(t+h)) &= \sum_{i=1}^{l} Ev^i(t, x(t)) + \int_t^{t+h} \sum_{i=1}^{l} ELv^i(s, x(s))\mathrm{d}s \\ &\leqslant \sum_{i=1}^{l} Ev^i(t, x(t)) + \int_t^{t+h} \beta(s) \sum_{i=1}^{l} Ev^i(s, x(s))\mathrm{d}s.\end{aligned}$$

Denote $m(t) = \sum_{i=1}^{l} Ev^i(t, x(t))$ for $t \in [\tau_{k-1}, \tau_k)$. Then

$$m(t+h) - m(t) \leqslant \int_t^{t+h} \beta(s) \sum_{i=1}^{l} Ev^i(s, x(s))\mathrm{d}s = \int_t^{t+h} \beta(s) m(s)\mathrm{d}s.$$

Therefore the upper Dini derivative of $m(t)$, denoted by $D^+m(t)$, satisfies

$$D^+m(t) \leqslant \beta(t) m(t).$$

By Theorem 1.4.1 in [19] we can prove that

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant \sum_{i=1}^{l} Ev^i(\tau_{k-1}, x(\tau_{k-1})) \exp\Big(\int_{\tau_{k-1}}^{t} \beta(s)\mathrm{d}s\Big).$$

That completes the proof.

Theorem 1. Let a_1, a_2, λ, a be positive numbers, integrable functions $\lambda_1(t), d_k(t) : [t_0, \infty) \to \mathbb{R}$. If there exists $v(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^l_+)$ such that

(i) for any $t \geqslant t_0, x \in \mathbb{R}^n$, $a_1|x(t)|^p \leqslant \sum_{i=1}^{l} v^i(t,x) \leqslant a_2|x(t)|^p$;
(ii) for all $k \in \mathbb{N}$, $\sum_{i=1}^{l} v^i(\tau_k, x(\tau_k)) \leqslant \sum_{i=1}^{l} d_k(\tau_k) v^i(\tau_k, x(\tau_k^-))$;
(iii) $\sum_{i=1}^{l} Lv^i(t,x) \leqslant (-\lambda + \lambda_1(t)) \sum_{i=1}^{l} v^i(t,x), t \neq \tau_k$;
(iv) $\prod_{j=1}^{\infty} d_j(\tau_j) < \infty$, $\int_{t_0}^{\infty} \lambda_1(s)\mathrm{d}s < \infty$,
then the zero solution of (1) is p-th exponentially stable and the decay rate is λ.

Proof. By condition (iv), there exists $M > 0$ such that $\prod_{j=1}^{\infty} d_j(\tau_j) \exp\left(\int_{t_0}^{\infty} \lambda_1(s)\mathrm{d}s\right) < M$.

For $t \in [t_0, \tau_1)$ by Lemma 1, conditions (i) and (iii), we have

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant a_2 E|x_0|^p \exp\left(-\lambda(t-t_0) + \int_{t_0}^{t} \lambda_1(s)\mathrm{d}s\right).$$

In the following we shall prove, for $t \in [\tau_{m-1}, \tau_m), m \in \mathbb{N}$, that

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant a_2 E|x_0|^p \prod_{j=1}^{m-1} d_j(\tau_j) \exp\left(-\lambda(t-t_0) + \int_{t_0}^{t} \lambda_1(s)\mathrm{d}s\right). \tag{2}$$

Suppose (2) holds for $t \in [\tau_{m-1}, \tau_m), m = k$, i.e.

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant a_2 E|x_0|^p \prod_{j=1}^{k-1} d_j(\tau_j) \exp\left(-\lambda(t-t_0) + \int_{t_0}^{t} \lambda_1(s)\mathrm{d}s\right),$$

and when $t = \tau_k$ by condition (ii),

$$\begin{aligned}\sum_{i=1}^{l} Ev^i(\tau_k, x(\tau_k)) &= \sum_{i=1}^{l} Ev^i(\tau_k, x(\tau_k^-) + I_k(x(\tau_k))) \\ &\leqslant \sum_{i=1}^{l} d_k(\tau_k) Ev^i(\tau_k^-, x(\tau_k^-)) \\ &\leqslant a_2 E|x_0|^p \prod_{j=1}^{k} d_j(\tau_j) \exp\left(-\lambda(\tau_k - t_0) + \int_{t_0}^{\tau_k} \lambda_1(s)\mathrm{d}s\right).\end{aligned}$$

Then, for $t \in [\tau_k, \tau_{k+1})$, by Lemma 1, we have

$$\begin{aligned}\sum_{i=1}^{l} Ev^i(t, x(t)) &\leqslant \sum_{i=1}^{l} Ev^i(\tau_k, x(\tau_k)) \exp\left(-\lambda(t-\tau_k) + \int_{\tau_k}^{t} \lambda_1(s)\mathrm{d}s\right) \\ &\leqslant a_2 E|x_0|^p \prod_{j=1}^{k} d_j(\tau_j) \exp\left(-\lambda(t-t_0) + \int_{t_0}^{t} \lambda_1(s)\mathrm{d}s\right).\end{aligned}$$

Through mathematical induction we can conclude that (2) holds for all $t \in [\tau_{m-1}, \tau_m), m \in \mathbb{N}$, that is,

$$\begin{aligned}\sum_{i=1}^{l} Ev^i(t,x) &\leqslant a_2 E|x_0|^p \prod_{j=1}^{m-1} d_j(\tau_j) \exp\left(-\lambda(t-t_0) + \int_{t_0}^{t} \lambda_1(s)\mathrm{d}s\right) \\ &\leqslant a_2 E|x_0|^p \prod_{j=1}^{\infty} d_j(\tau_j) \exp\left(-\lambda(t-t_0) + \int_{t_0}^{\infty} \lambda_1(s)\mathrm{d}s\right).\end{aligned}$$

Note that even if there doesn't exist k such that $t \in [\tau_{k-1}, \tau_k)$, that is, $\lim_{k\to\infty} \tau_k = \tau < \infty$, the relation above still holds. Then from conditions (i) and (iv) it follows that

$$E|x(t)|^p < ME|x_0|^p \exp(-\lambda(t-t_0)),$$

by which we can conclude that the solution of (1) is globally p-th exponentially stable in the sense of [2], and the decay rate is λ. Therefore the solution of (1) is p-th exponentially stable in the sense of definition 1 and the decay rate is λ. That completes the proof.

Remark 1 Theorem 1 can also be applied to prove the uniform p-th moment stability investigated in [15]. Note that the conditions of Theorem 3 in [15] when dealing with (1) is stronger than the conditions of Theorem 1. And especially when $I_k(x) = 0$, Theorem 1 in [16] and Theorem 8 in [18] can be obtained directly from Theorem 1.

Theorem 2. Let a_1, a_2, r, a, c be positive numbers, $d_k(t) : [t_0, \infty) \to \mathbb{R}$. If there exists $v(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^l_+)$ such that conditions (i) and (ii) of Theorem 1 hold, while conditions (iii) and (iv) are replaced by

(iii) for $t \neq \tau_k$, $\sum_{i=1}^l Lv^i(t,x) \leqslant c\sum_{i=1}^l v^i(t,x)$;

(iv) $d_k(\tau_k), k \in \mathbb{N}$ satisfies $\ln[d_k(\tau_k)] \leqslant -(a+c)r$ with $0 \leqslant \tau_k - \tau_{k-1} \leqslant r$, then the zero solution of (1) is p-th exponentially stable with the decay rate a.

Proof. For any $\varepsilon > 0$, choose $\delta > 0$ such that

$$\delta < a_1/a_2 \varepsilon \mathrm{e}^{-(a+c)r} \qquad \text{and} \qquad E|x_0|^p < \delta.$$

Note that, by Lemma 1 and condition (iii), for $t \in [\tau_{k-1}, \tau_k)$, we have

$$\sum_{i=1}^l Ev^i(t,x) \leqslant \sum_{i=1}^l Ev^i(\tau_{k-1}, x(\tau_{k-1}))\mathrm{e}^{c(t-t_{k-1})}. \tag{3}$$

We shall prove, for $t \in [\tau_{m-1}, \tau_m), m \in \mathbb{N}$

$$\sum_{i=1}^l Ev^i(t,x) \leqslant a_1\varepsilon \mathrm{e}^{-(a+c)kr+c(t-t_0)}, \tag{4}$$

$$E|x(t)|^p < \varepsilon \mathrm{e}^{-a(t-t_0)}. \tag{5}$$

By (3) and condition (i), for $t \in [t_0, \tau_1)$, we have

$$\sum_{i=1}^l Ev^i(t,x) \leqslant \sum_{i=1}^l Ev^i(t_0, x_0)\mathrm{e}^{c(t-t_0)} \leqslant a_2 E|x_0|^p \mathrm{e}^{c(t-t_0)} \leqslant a_1\varepsilon \mathrm{e}^{-(a+c)r+c(t-t_0)}, \tag{6}$$

and thus for $t \in [t_0, \tau_1)$, $E|x(t)|^p \leqslant \frac{1}{a_1}\sum_{i=1}^l Ev^i(t,x) \leqslant \varepsilon \mathrm{e}^{-(a+c)r}\mathrm{e}^{c(t-t_0)} \leqslant \varepsilon \mathrm{e}^{-a(t-t_0)}$. For $t \in [\tau_{m-1}, \tau_m)$, $m = k$, suppose (4) and (5) hold, that is,

$$\sum_{i=1}^l Ev^i(t,x) \leqslant a_1\varepsilon \mathrm{e}^{-(a+c)kr+c(t-t_0)},$$
$$E|x(t)|^p < \varepsilon \mathrm{e}^{-a(t-t_0)}.$$

Then when $t = \tau_k$

$$\begin{aligned}\sum_{i=1}^l Ev^i(\tau_k, x(\tau_k)) &\leqslant \sum_{i=1}^l d_k Ev^i(\tau_k^-, x(\tau_k^-)) \leqslant \mathrm{e}^{-(a+c)r} \cdot a_1\varepsilon \mathrm{e}^{-(a+c)kr+c(\tau_k-t_0)} \\ &= a_1\varepsilon \mathrm{e}^{-(a+c)(k+1)r+c(\tau_k-t_0)}.\end{aligned}$$

Then for $t \in [\tau_k, \tau_{k+1})$ we have $\sum_{i=1}^l Ev^i(t, x(t)) \leqslant \sum_{i=1}^l Ev^i(\tau_k, x(\tau_k))\mathrm{e}^{c(t-\tau_k)} \leqslant a_1\varepsilon \mathrm{e}^{-(a+c)(k+1)r+c(t-t_0)}$ and $E|x(t)|^p < \varepsilon \mathrm{e}^{-a(t-t_0)}$.

By a similar proof and the mathematical induction we can conclude that for all $t \in [t_0, \infty)$, (4) and (5) hold; that is, the solution of (1) is p-th exponential stable with decay rate a. And that completes the proof.

Remark 2. Note that $\sum_{i=1}^{l} Lv^i(t,x) \leqslant c, c > 0$ and only the conditions used in the globally p-th moment exponential stability are not enough to reduce the exponent to the decay rate a. Therefore, unlike the proof of globally p-th moment exponential stability it is especially important to choose the range δ of the initial value $E|x_0|^p$. Due to the selective condition of the initial value, the Definition 1 is local. Although this locally p-th moment exponential stability imposes condition of the initial value, it, at the same time, relaxes requirements of other parameters, such as the length of impulsive interval. See Example 4 for example.

Theorem 3. Let a_1, a_2 be positive numbers, $d_k(t) : [t_0, \infty) \to \mathbb{R}$. If there exists $v(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^l_+)$ such that conditions (i) and (ii) of Theorem 1 hold, while conditions (iii) and (iv) are replaced by

(iii) $\sum_{i=1}^{l} Lv^i(t,x) \leqslant 0$;

(iv) $d_k(\tau_k), k \in \mathbb{N}$ satisfies that $\ln d_k(\tau_k) \leqslant -a(\tau_{k+1} - \tau_k)$,

then the trivial solution of (1) is p-th moment exponentially stable with decay rate a.

Proof. For any given $\varepsilon > 0$, choose $\delta > 0$ such that

$$\delta < a_1/a_2 \varepsilon \mathrm{e}^{-a(\tau_1 - t_0)}, \qquad E|x_0|^p < \delta. \tag{7}$$

By Lemma 1 and condition (iii) for $t \in [\tau_{k-1}, \tau_k), k \in \mathbb{N}$ we can obtain

$$\sum_{i=1}^{l} Ev^i(t, x(t)) \leqslant \sum_{i=1}^{l} Ev^i(\tau_{k-1}, x(\tau_{k-1})). \tag{8}$$

We will prove, for $t \in [\tau_{k-1}, \tau_k), k \in \mathbb{N}$, the following inequalities hold:

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant a_1 \varepsilon \mathrm{e}^{-a(t-t_0)}, \tag{9}$$

$$E|x(t)|^p < \varepsilon \mathrm{e}^{-a(t-t_0)}. \tag{10}$$

For $t \in [t_0, \tau_1)$, by (8) and condition (i), we have

$$\begin{aligned} \sum_{i=1}^{l} Ev^i(t,x) &\leqslant \sum_{i=1}^{l} Ev^i(t_0, x(t_0)) \leqslant a_2 E|x_0|^p \leqslant a_1 \varepsilon \mathrm{e}^{-a(\tau_1 - t_0)} \\ &\leqslant a_1 \varepsilon \mathrm{e}^{-a(t-t_0)}, \end{aligned}$$

and $E|x(t)|^p < \varepsilon e^{-a(t-t_0)}$. Suppose (9) and (10) hold for all $t \in [\tau_{m-1}, \tau_m), m = k$. Then when $t = \tau_k$

$$\begin{aligned} \sum_{i=1}^{l} Ev^i(\tau_k, x(\tau_k)) &\leqslant \sum_{i=1}^{l} d_k(\tau_k) Ev^i(\tau_k^-, x(\tau_k^-)) \leqslant a_1 \mathrm{e}^{-a(\tau_{k+1} - \tau_k)} \varepsilon \mathrm{e}^{-a(\tau_k - t_0)} \\ &< a_1 \varepsilon \mathrm{e}^{-a(\tau_{k+1} - t_0)}. \end{aligned}$$

Thus for $t \in [\tau_k, \tau_{k+1})$, we have

$$\sum_{i=1}^{l} Ev^i(t,x) \leqslant \sum_{i=1}^{l} Ev^i(\tau_k, x(\tau_k)) \leqslant a_1 \varepsilon \mathrm{e}^{-a(t-t_0)}.$$

By condition (i), we have

$$E|x(t)|^p \leqslant \varepsilon \mathrm{e}^{-a(t-t_0)},$$

which implies that (9) and (10) hold for $m = k + 1$. Therefore the desired result can be obtained. And that completes the proof.

Remark 3. Note that the bounded condition on $\tau_k - \tau_{k-1}$ in Theorem 2 is removed in Theorem 3.

To formulate the next assertion, consider the one-dimensional system in the form of (1) with $n = m = 1$. In the following theorem, the usual condition on Lv such as condition (iii) in Theorems 1–3, respectively, is replaced by a condition on the drift and diffusion coefficients.

Theorem 4. Let $a_1, a_2, a, d_k, k \in \mathbb{N}$ be positive numbers. If there exists $v(t,x) \in C^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}_+)$ such that

(i) $a_1 E|x(t)|^p \leqslant Ev(t,x(t)) \leqslant a_2 E|x(t)|^p$;

(ii) $v(\tau_k, x(\tau_k)) \leqslant d_k v(\tau_k, x(\tau_k^-))$;

(iii) the function f, g in (1) are assumed to satisfy $|f(t,x)| + |g^2(t,x)| \leqslant L|x|$;

(iv) $d_k, k \in \mathbb{N}$ satisfies the condition that $\ln d_k \leqslant -(a + \frac{1}{2}L_1)(\tau_k - \tau_{k-1}) + \ln \frac{a_1}{a_2}$ for all $k \in \mathbb{N}$, where $L_1 = 2Lp + p|2p-1|L^2$,

then the trivial solution of (1) is p-th moment exponentially stable with decay rate a.

Proof. For any $\varepsilon > 0$, choose a $\delta > 0$ such that

$$\delta < a_1/a_2 \varepsilon \mathrm{e}^{-a(\tau_1 - t_0)} \quad \text{and} \quad (E|x_0|^p)^2 \leqslant E|x_0|^{2p} < \delta^2.$$

By Itô formula, we have

$$\begin{aligned} \mathrm{d}x^{2p} &= 2p\, x^{2p-1}\mathrm{d}x + \frac{1}{2}2p\,(2p-1)x^{2p-2}(\mathrm{d}x)^2 \\ &= (2p\, x^{2p-1}f(t,x) + p\,(2p-1)g^2(t,x)x^{2p-2})\mathrm{d}t + 2p\, x^{2p-1}g(t,x)\mathrm{d}B, \end{aligned} \tag{11}$$

and rewriting (11) for $t \in [\tau_{k-1}, \tau_k)$ in the integral form, we can easily obtain

$$\begin{aligned} x^{2p}(t) = x^{2p}(\tau_{k-1}) &+ \int_{\tau_{k-1}}^t (2p\, x^{2p-1}f(s,x) + p\,(2p-1)g(s,x)x^{2p-2}g(s,x))\mathrm{d}s \\ &+ \int_{\tau_{k-1}}^t 2p\, x^{2p-1}g(s,x)\mathrm{d}B. \end{aligned} \tag{12}$$

Taking expectation of (12), by condition (iii), we have

$$\begin{aligned} E|x(t)|^{2p} &= E|x(\tau_{k-1})|^{2p} + E\int_{\tau_{k-1}}^t (2p\, x^{2p-1}f(s,x) + p\,(2p-1)g(s,x)x^{2p-2}g(s,x))\mathrm{d}s \\ &\leqslant E|x(\tau_{k-1})|^{2p} + \int_{\tau_{k-1}}^t L_1 E|x(s)|^{2p}\mathrm{d}s. \end{aligned} \tag{13}$$

It follows directly from (13) and Gronwall's inequality that for $t \in [\tau_{k-1}, \tau_k), k \in \mathbb{N}$

$$E|x(t)|^{2p} < E|x(\tau_{k-1})|^{2p}\mathrm{e}^{L_1(t-\tau_{k-1})}. \tag{14}$$

Assume $\tau_0 = t_0$. Then we shall prove, for $k \in \mathbb{N}$

$$\begin{aligned} E|x(t)|^{2p} &\leqslant \prod_{i=1}^{k-1}\left(d_i\frac{a_2}{a_1}\right)^2 \delta^2 \mathrm{e}^{L_1(t-t_0)}, t \in [\tau_{k-1}, \tau_k), \\ E|x(\tau_k)|^{2p} &\leqslant \prod_{i=1}^{k}\left(d_i\frac{a_2}{a_1}\right)^2 \delta^2 \mathrm{e}^{L_1(\tau_k-t_0)} \quad t = \tau_k. \end{aligned} \tag{15}$$

For $t \in [t_0, \tau_1)$, by (14) we can easily obtain

$$E|x(t)|^{2p} \leqslant E|x_0|^{2p}\mathrm{e}^{L_1(t-t_0)} < \delta^2\mathrm{e}^{L_1(t-t_0)}$$

and by conditions (i) and (iii), we have

$$E|x(\tau_1)|^{2p} \leqslant E\left(\frac{1}{a_1}v(\tau_1, x(\tau_1))\right)^2 \leqslant E\left(\frac{d_1}{a_1}v(\tau_1, x(\tau_1))\right)^2 \leqslant E\left(d_1\frac{a_2}{a_1}|x(\tau_1^-)|^p\right)^2$$

$$\leqslant \left(d_1 \frac{a_2}{a_1}\right)^2 E|x_0|^{2p} \mathrm{e}^{L_1(\tau_1 - t_0)} \leqslant \left(d_1 \frac{a_2}{a_1}\right)^2 \delta^2 \mathrm{e}^{L_1(\tau_1 - t_0)}.$$

Suppose (15) holds for $m = k$. Then we will prove (15) still holds for $m = k+1$. For $t \in [\tau_k, \tau_{k+1})$, we have

$$E|x(t)|^{2p} \leqslant E|x(\tau_k)|^{2p} \mathrm{e}^{L_1(t-\tau_k)} \leqslant \prod_{i=1}^{k} \left(d_i \frac{a_2}{a_1}\right)^2 \delta^2 \mathrm{e}^{L_1(\tau_k - t_0)} \cdot \mathrm{e}^{L_1(t-\tau_k)}$$

and when $t = \tau_{k+1}$

$$E|x(\tau_{k+1})|^{2p} \leqslant E\left(\frac{1}{a_1} v(\tau_{k+1}, x(\tau_{k+1}))\right)^2 \leqslant E\left(\frac{d_{k+1}}{a_1} v(\tau_{k+1}, x(\tau_{k+1}^-))\right)^2$$

$$\leqslant \left(\frac{a_2}{a_1} d_{k+1}\right)^2 E|x(\tau_{k+1}^-)|^{2p} \leqslant \prod_{i=1}^{k+1} \left(d_i \frac{a_2}{a_1}\right)^2 \delta^2 \mathrm{e}^{L_1(t_{k+1} - t_0)}.$$

By the induction we can conclude that (15) holds for all $k \in \mathbb{N}$. Since $(E|x(t)|^p)^2 \leqslant E|x(t)|^{2p}$, we have $E|x(t)|^p < \prod_{j=1}^{k} d_j \frac{a_2}{a_1} \delta \exp^{\frac{1}{2} L_1(t-t_0)}$. Then if $d_k, k \in \mathbb{N}$ satisfies condition (iv), for $t \in [\tau_{k-1}, \tau_k)$, we have $E|x(t)|^p \leqslant \varepsilon \mathrm{e}^{-a(\tau_k - t_0)} < \varepsilon \mathrm{e}^{-a(t-t_0)}$. And that completes the proof.

Remark 4.

1) In Theorem 4 the usual condition on $Lv(t,x)$ is replaced by condition (iii). And if $f(t,x), g^2(t,x)$ satisfy the global Lipschitz condition,

$$|f(t,x) - f(t,y)| + |g^2(t,x) - g^2(t,y)| \leqslant M|x-y|,$$

then condition (iii) follows directly.

2) Here we must remind the readers of a detail in the proof of Theorem 4. Note that we carry out Itô formula on the $x^{2p}(t)$ in (11), not $x^p(t)$. If $x^p(t)$ is chosen to take the place of $x^{2p}(t)$, then we cannot obtain the inequality without the item $-\mathrm{d}B$ like (13).

4 Numerical examples

To illustrate the validity of our main results we give the following numerical examples.

Example 1. Consider the following stochastic impulsive differential equation:

$$\begin{cases} \mathrm{d}x_1 = (-\mu_1 + m_1 \mathrm{e}^{-t}) x_1 \mathrm{d}t + \sqrt{\mu_1} x_2 \mathrm{d}B, t \neq \tau_k, \\ \mathrm{d}x_2 = (-\mu_2 + m_2 \mathrm{e}^{-t}) x_2 \mathrm{d}t + \sqrt{\mu_2} x_1 \mathrm{d}B, t \neq \tau_k, \\ x_1(\tau_k) = c_1(\tau_k) x_1(\tau_k), t = \tau_k, \\ x_2(\tau_k) = c_2(\tau_k) x_2(\tau_k), t = \tau_k, \end{cases} \tag{16}$$

where $\mu_1, \mu_2 > 0$, $c(\tau_k) = \max\{c_1^2(\tau_k), c_2^2(\tau_k)\} \leqslant h \in (0,1)$. Taking $v(t,x) = \binom{x_1^2}{x_2^2}$ and applying infinitesimal generator, we obtain

$$Lv^1(t,x) + Lv^2(t,x) \leqslant (-\mu + m\mathrm{e}^{-t})(x_1^2 + x_1^2),$$

and

$$Ev^1(\tau_k, x(\tau_k)) + Ev^2(\tau_k, x(\tau_k)) \leqslant c(\tau_k)(Ev^1(\tau_k^-, x(\tau_k^-)) + Ev^2(\tau_k^-, x(\tau_k^-))),$$

where $m = 2\max\{m_1, m_2\}$, $\mu = \min\{\mu_1, \mu_2\}$. $c(\tau_k) \leqslant h \in (0,1)$ implies $\prod_{j=1}^{\infty} c(\tau_k) < \infty$. Then by Theorem 1, (16) is exponentially stable in mean square with decay rate μ.

Example 2. Consider the following stochastic impulsive differential equation:

$$\begin{cases} \mathrm{d}x_1 = -x_1 \mathrm{e}^{\frac{1}{t+1}} \mathrm{d}t + \sqrt{2} x_1 \sin t \mathrm{e}^{-t(x_1^2 + x_2^2)} \mathrm{d}B, \\ \mathrm{d}x_2 = -x_2 \mathrm{e}^{\frac{1}{t+1}} \mathrm{d}t + \sqrt{2} x_2 \sin t \mathrm{e}^{-t(x_1^2 + x_2^2)} \mathrm{d}B, \end{cases} \tag{17}$$

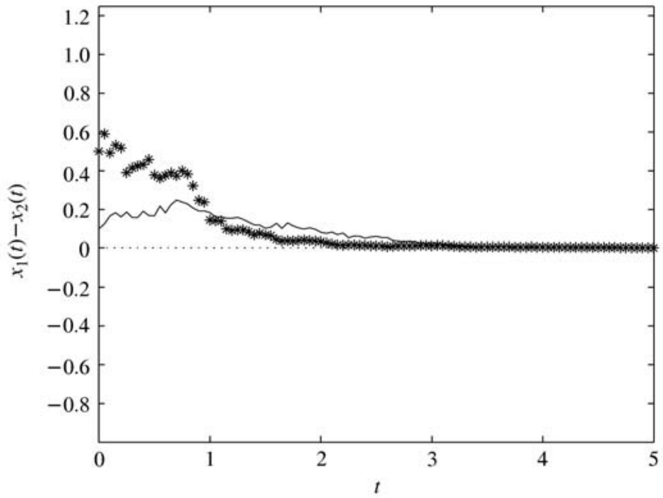

Figure 1 Stability of (1) under impulsive control.

with impulse effect

$$x(t) = w_k(t)x(t^-), \quad t = k, k \in \mathbb{N}.$$

Taking $v(t,x) = (\begin{smallmatrix} x_1^2 \\ x_2^2 \end{smallmatrix})$, and applying infinitesimal generator along with (17) we have

$$Lv(t,x) = \begin{pmatrix} 2x_1^2 \sin^2 t\mathrm{e}^{-2t(x_1^2+x_2^2)} - 2x_1^2\mathrm{e}^{\frac{1}{t+1}} \\ 2x_2^2 \sin^2 t\mathrm{e}^{-2t(x_1^2+x_2^2)} - 2x_2^2\mathrm{e}^{\frac{1}{t+1}} \end{pmatrix} \leqslant c \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix},$$

with $c = 2$, which implies

$$Lv^1(t,x) + Lv^2(t,x) \leqslant 2(v^1(t,x) + v^2(t,x)).$$

Let $a > 0$ be given. If w_k satisfies

$$\ln|w_k(k)| \leqslant -\frac{a+2}{2},$$

then by Theorem 2 the trivial solution of (17) is p-th exponentially stable with decay rate a.

Example 3. Consider the following stochastic impulsive differential equation:

$$\mathrm{d}x(t) = \sin x(t)\mathrm{d}t + x(t)\mathrm{d}B, \tag{18}$$

with impulse effect

$$x(t) = e^{\gamma_k}x(t^-), \quad t = \tau_k, k \in \mathbb{N}.$$

Taking $v(t,x) = x^2$ we have $Ex^2 \leqslant Ev(t,x) \leqslant 2Ex^2$, with $a_1 = 1, a_2 = 2$. Using the coefficients defined in (18) we can choose the Lyapunov constant $L = 1$. When $p = 2$ if γ_k satisfies

$$\gamma_k \leqslant -\frac{a+L_1}{2}(\tau_k - \tau_{k-1}) + \ln\frac{1}{2} = -(0.5a+10)(\tau_k - \tau_{k-1}) + ln\frac{1}{2},$$

where L_1 is defined by condition (iv) in Theorem 4, then by Theorem 4 the trivial solution of (16) is exponentially stable in mean square with decay rate a.

Example 4. Apply the Theorem 2 to the example 4.1 in [17] as follows:

$$\mathrm{d}x = f(t,x)\mathrm{d}t + \varepsilon g(t,x)\mathrm{d}w(t), t \geqslant t_0, \tag{19}$$

where $f(t,x) = (ax_1 + 2x_2(x_1^2 + x_2^2), ax_2 - x_1(x_1^2 + x_2^2))^T$, $g(t,x) = (x_1, -x_2)^T$, $a = -1, \varepsilon = \sqrt{3}$. Choose the same Lyapunov functions $v(x) = 1/2(x_1^2 + x_2^2)$. Since $Lv(t,x) = (a+\varepsilon^2)v > 0$, (19) is unstable. Let the impulsive control be $I_k(x) = (-1 + e^{-2-1/(k+1)})x$ as in [17] with the longer impulsive interval $\tau_k - \tau_k = 2$. Then (19) with these impulses is exponentially stable according to Theorem 2 (Figure 1).

5 Conclusions

The p-th exponential stability of stochastic impulsive differential equations has been investigated. Some sufficient conditions are obtained according to the frequency of the impulse. In addition, the usual condition of the growth rate of Lyapunov functions is replaced by the condition of the drift and diffusion coefficients to study the stability. Some numerical examples have been given to show the feasibility our results.

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References

1 Oksendal B. Stochastic Differential Equations: An Introduction with Applications. Berlin: Springer, 2003
2 Mao X. Stochastic Differential Equations and Their Applications. Chichester: Horwood Publishing Ltd, 1997
3 Chen H F. Stochastic approximation with state-dependent noise. Sci China, Ser E-Tech Sci, 2000, 43: 531–541
4 Guan Z H, Chen W H, Xu J X. Delay-dependent stability and stabilizability of uncertain jump bilinear stochastic systems with mode-dependent time-delays. Int J Syst Sci, 2005, 36: 275–285
5 Hou Z T, Luo J W, Shi P, et al. Stochastic stability of Ito differential equations with semi-markovian jump parameters. IEEE Trans Autom Control, 2006, 51: 1383–1387
6 Chatterjee D, Liberzon D. On stability of randomly switched nonlinear systems. IEEE Trans Autom Control, 2007, 52: 2390–2394
7 Zhao P. Practical stability, controllability and optimal control of stochastic Markovian jump systems with time-delays. Automatica, 2008, 44: 3120–3125
8 Mao X R, Lam J, Huang L R. Stabilisation of hybrid stochastic differential equations by delay feedback control. Syst Control Lett, 2008, 57: 927–935
9 Huang L R, Deng F Q. Razumikhin-type theorems on stability of neutral stochastic functional differential equations. IEEE Trans Autom Control, 2008, 53: 1718–1723
10 Lakshmikantham V, Bainov D D, Simeonov P S. Theory of Impulsive Differential Equations. Singapore: World Scientific, 1989
11 Guan Z H, Hill D J, Shen X. Hybrid impulsive and switching systems and application to control and synchronization. IEEE Trans Autom Control, 2005, 50: 1058–1062
12 Chen W H, Zheng W X. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. Automatica, 2009, 45: 1481–1488
13 Xu S Y, Chen T W. Robust H_∞ filtering for uncertain impulsive stochastic systems under sampled measurements. Automatica, 2003, 39: 509–516
14 Zhang H, Guan Z H, Feng G. Reliable dissipative control for stochastic impulsive systems. Automatica, 2008, 44: 1004–1010
15 Wu H J, Sun J T. p-moment stability of stochastic differential equations with impulsive jump and Markovian switching. Automatica, 2006, 42: 1753–1759
16 Xu W, Niu Y J, Rong H W, et al. p-moment stability of stochastic impulsive differential equations and its application in impulsive control. Sci China Ser. E-Tech Sci, 2009, 52: 782–786
17 Liu B. Stability of solutions for stochastic impulsive systems via comparison approach. IEEE Trans Autom Control, 2008, 53: 2128–2133
18 Baker C T H, Buckwar E. Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations. J Comput Appl Mathematics, 2005, 184: 404–427
19 Lakshmikantham V, Leela S. Differential and Integral Inequalities of Theory and Applications. New York: Academic Press, 1969