带有线性外力场的双曲平均曲率流 Cauchy 问题经典解的生命跨度

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摘要 本文提出并研究带有线性外力场的双曲平均曲率流, 通过凸曲线的支撑函数, 导出一个双曲型 Monge-Ampère 方程并将其转化成 Riemann 不变量满足的拟线性双曲方程组. 利用拟线性双曲方程组 Cauchy 问题的局部解理论, 讨论带有线性外力场的双曲平均曲率流 Cauchy 问题经典解的生命跨度 (即局部解存在的最大时间区间).

关键词 线性外力场 双曲平均曲率流 双曲型 Monge-Ampère 方程 Riemann 不变量 生命跨度

MSC (2010) 主题分类 58J45, 58J47, 53C44, 35L70, 35L45

1 引言

本文研究带有外力场的双曲平均曲率流作用下的凸曲线的演化. 更准确地讲, 考虑初值问题

$$\begin{cases} \frac{\partial^2 \gamma}{\partial t^2}(u, t) = (k(u, t) - \langle c\gamma, \vec{N} \rangle) \vec{N}(u, t) - \nabla \rho(u, t), & \forall (u, t) \in I \times [0, T), \\ \gamma(u, 0) = \gamma_0(u), \\ \frac{\partial \gamma}{\partial t}(u, 0) = f(u) \vec{N}_0, \end{cases} \quad (1.1)$$

其中 k 是曲线 $\gamma(u, t)$ 的平均曲率, $\vec{N}$ 是曲线 $\gamma(u, t)$ 的单位内法向量, γ_0 表示光滑严格凸曲线, $f(u)$ (≥ 0) 和 $\vec{N}_0$ 分别是 γ_0 的初始速度和初始内法向量, $\vec{T}$ 是单位切向量, s 是弧长参数, $\nabla \rho$ 定义为

$$\nabla \rho \triangleq \left(\frac{\partial^2 \gamma}{\partial s \partial t}, \frac{\partial \gamma}{\partial t} \right) \vec{T}.$$

最近, Yau^[1] 提出了与振动膜或曲面运动相关的方程

$$\frac{\partial^2 X}{\partial t^2} = H \vec{n}, \quad (1.2)$$

其中 H 是平均曲率, $\vec{n}$ 是曲面的单位内法向量, 并且给出了超曲面的整体时间行为. Gurtin 和 Podio-Guidugli^[2] 发展了平面曲线演化的双曲理论. Rostein 等人^[3] 进一步研究了平均曲率流方程的双曲理论, 并给出了关于闭多项式曲线运动的一个晶体算法. He 等人^[4] 首次从几何的观点介绍了双曲平均

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曲率流. Lefloch 和 Smoczyk^[5] 研究了一类双曲型几何演化方程. 最近, 当 $c = 0$ 时, 双曲曲线收缩问题被 Kong 等人^[6] 研究. 在几何上, 这样的流使得凸性在发展运动中保持并且凸初始曲线在满足一定的初始速度条件下, 在有限时间内要么收缩成一点, 要么收缩为一个连续但分段光滑的闭的弱凸曲线. Kong 和 Wang^[7] 进一步着重研究了双曲平均曲率流作用下平面曲线运动奇性的形成.

与前面研究的双曲平均曲率流相比, Chou 和 Wo^[8] 提出了一个新的凸超曲面的双曲曲率流. 当 Gauss 曲率涉及时, 这个流最合适. 在这个流下利用超曲面图满足的方程提供了一类新的非线性 Euclid 不变量双曲方程. 最近, Notz^[9] 在他的博士论文中介绍并研究了一新的几何方程, 描述 Riemann 流形中闭超曲面的运动. 如果曲面是球形的, 这个方程被考虑为一个移动肥皂泡的非理想化数学模型, 可通过一个作用积分的 Euclid-Lagrange 方程. 另外就动能来讲, 这个作用积分包含了依赖于所围体积的曲面张量和内压力. 相应的 Euclid-Lagrange 方程是一个拟线性退化的二阶双曲型偏微分方程, 描述了曲面的运动. 作者证明了短时间存在性定理, 并且给定一个充分条件保证解能延拓到大的时间区间上.

最近, 文献 [10, 11] 根据曲率估计研究了带有外力场的平均曲率流光滑解的长时间存在性. Li 等人^[12] 研究了 Euclid 空间带有外力场的平均曲率流, 他们证明了不同的外力项导致不同的收敛行为. 然而, 据我们所知, 带有外力场的双曲平均曲率流至今未曾有人研究.

本文主要结果是下面的 (两个) 定理.

定理 1.1 (局部存在唯一性) 假设 γ_0 是光滑严格凸闭曲线, $f(u)$ 是 S^1 上的光滑函数, 则存在一个正数 T 和一簇严格凸闭曲线 $\gamma(\cdot, t)$ ($t \in [0, T)$) 使得 $\gamma(\cdot, t)$ 满足 (1.1).

下面的定理给出初值问题经典解的生命跨度.

定理 1.2 假设 γ_0 是严格凸闭曲线, $f(u)$ 是 S^1 上的光滑函数, 则局部解存在区间的下界 δ_* 为

$$\delta_* = \min \left\{ \frac{\|\phi\|}{A_0 - \|\phi\|}, \frac{A_0 + A_1 \|\dot{\phi}\|}{B_0 + B_1 \|\dot{\phi}\| + B_2 \|\dot{\phi}\|^2}, \frac{1}{C_0 + C_1 \|\dot{\phi}\|} \right\}, \quad (1.3)$$

其中

$$\begin{aligned} \phi &= (\phi_1, \dots, \phi_5)^T = \left(\frac{1 - \tilde{f}'(\theta)}{g''(\theta) + g(\theta)}, \frac{-1 - \tilde{f}'(\theta)}{g''(\theta) + g(\theta)}, g(\theta), \tilde{f}(\theta), g'(\theta) \right)^T, \\ A_0(\|\phi\|) &= (8\|\phi\|^3 + 4\max\{|c|, 1\}\|\phi\|^2 + 2|c|\|\phi\| + 3\|\phi\| + 1), \quad A_1(\|\phi\|) = 2\|\phi\| + 1, \\ A_2(\|\phi\|) &= A_0 - \|\phi\|, \quad A_3(\|\phi\|) = 48\|\phi\|^2 + 16\max\{|c|, 1\}\|\phi\| + 4|c| + 4, \\ B_0(\|\phi\|) &= 2A_0A_1A_3 + A_2, \quad B_1(\|\phi\|) = 6A_0A_1 + 2A_1^2A_3, \quad B_2(\|\phi\|) = 6A_1^2, \\ C_0(\|\phi\|) &= 4(1 + 2\|\phi\|)^2(48\|\phi\|^2 + 16\max\{|c|, 1\}\|\phi\| + 4|c| + 5) \\ &\quad \times (12\|\phi\|^2 + 4\max\{|c|, 1\}\|\phi\| + |c| + 1), \\ C_1(\|\phi\|) &= 16(1 + 2\|\phi\|)^2(12\|\phi\|^2 + 4\max\{|c|, 1\}\|\phi\| + |c| + 1). \end{aligned}$$

本文组织如下: 第 2 节导出一个双曲型 Monge-Ampère 方程; 第 3 节导出双曲型 Monge-Ampère 方程的 Riemann 不变量满足的方程组, 得到解的局部存在唯一性; 第 4 节利用拟线性双曲方程组 Cauchy 问题的局部解理论^[13], 主要结果即定理 1.2 得到证明.

2 双曲型 Monge-Ampère 方程

在研究凸曲线的运动中, 利用支撑函数描述该流比图更好. 设 θ 是严格凸曲线 $F: I \rightarrow \mathbb{R}^2$ 的单位

外法向角. 因此,

$$\vec{N} = (-\cos \theta, -\sin \theta), \quad \vec{T} = (-\sin \theta, \cos \theta),$$

由 Frenet 公式, 我们有 $\frac{\partial \theta}{\partial s} = k$.

假设 $\gamma(u, t) : I \times [0, T) \rightarrow \mathbb{R}^2$ 是一簇满足带有外力场的双曲平均曲率流 (1.1) 的光滑闭凸曲线. 我们用法向角参数化任意闭凸曲线 $\gamma(\cdot, t)$, 即取

$$\tilde{\gamma}(\theta, \tau) = \gamma(u(\theta, \tau), t(\theta, \tau)),$$

其中 $t(\theta, \tau) = \tau$. 由链式法则,

$$\frac{\partial \tilde{\gamma}}{\partial \tau} = \frac{\partial \gamma}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial \gamma}{\partial t},$$

并且

$$\frac{\partial^2 \tilde{\gamma}}{\partial \tau^2} = \frac{\partial \gamma}{\partial u} \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 \gamma}{\partial u^2} \left(\frac{\partial u}{\partial \tau} \right)^2 + 2 \frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 \gamma}{\partial t^2}.$$

γ 的支撑函数为

$$h(\theta, \tau) = (\tilde{\gamma}(\theta, \tau), \vec{N}) = (\tilde{\gamma}(\theta, \tau), (\cos \theta, \sin \theta)) = x(\theta, \tau) \cos \theta + y(\theta, \tau) \sin \theta.$$

我们有

$$h_\theta(\theta, \tau) = -x(\theta, \tau) \sin \theta + y(\theta, \tau) \cos \theta.$$

因此, 曲线 γ 可由 h 和 h_θ ,

$$\begin{cases} x = h \cos \theta - h_\theta \sin \theta, \\ y = h \sin \theta + h_\theta \cos \theta. \end{cases}$$

这样, 曲线的所有几何量都可以由支撑函数给出. 特别地, 曲率为

$$k = \frac{1}{h_{\theta\theta} + h}.$$

我们知道支撑函数 $h(\theta, \tau)$ 满足

$$h_\tau = \left(\frac{\partial \tilde{\gamma}}{\partial \tau}, -\vec{N} \right) = \left(\frac{\partial \gamma}{\partial u} \frac{\partial u}{\partial \tau} + \frac{\partial \gamma}{\partial t}, -\vec{N} \right) = \left(\frac{\partial F}{\partial t}, -\vec{N} \right),$$

并且,

$$\begin{aligned} h_{\tau\tau} &= \left(\frac{\partial^2 \tilde{\gamma}}{\partial \tau^2}, -\vec{N} \right) + \left(\frac{\partial \tilde{\gamma}}{\partial \tau}, -\frac{\partial \vec{N}}{\partial \tau} \right) \\ &= \left(\frac{\partial \gamma}{\partial u} \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 \gamma}{\partial u^2} \left(\frac{\partial u}{\partial \tau} \right)^2 + 2 \frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 \gamma}{\partial t^2}, -\vec{N} \right) \\ &= \left(\frac{\partial^2 \gamma}{\partial u^2} \left(\frac{\partial u}{\partial \tau} \right)^2 + \frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau}, -\vec{N} \right) + \left(\frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 \gamma}{\partial t^2}, -\vec{N} \right) \\ &= \left(\left(\frac{\partial \gamma}{\partial u} \right)_\tau, -\vec{N} \right) \frac{\partial u}{\partial \tau} + \left(\frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau} + \frac{\partial^2 \gamma}{\partial t^2}, -\vec{N} \right) \\ &= \left(\frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau}, -\vec{N} \right) - k - ch. \end{aligned}$$

根据法向演化曲线,

$$\left(\frac{\partial \gamma}{\partial t}, \vec{T}\right) \equiv 0, \quad t \in [0, T).$$

由

$$h_\tau = \left(\frac{\partial \gamma}{\partial t}, -\vec{N}\right),$$

我们有

$$\begin{aligned} h_{\theta\tau} &= \left(\frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \theta}, -\vec{N}\right) + \left(\frac{\partial \gamma}{\partial t}, -\frac{\partial \vec{N}}{\partial \theta}\right) \\ &= \left(\frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \theta}, -\vec{N}\right) = \frac{1}{\partial \theta / \partial u} \left(\frac{\partial^2 \gamma}{\partial u \partial t}, -\vec{N}\right) \\ &= \frac{1}{kv} \left(\frac{\partial^2 \gamma}{\partial u \partial t}, -\vec{N}\right). \end{aligned}$$

另一方面, $h_\theta = (\tilde{\gamma}, \vec{T})$, 则

$$h_{\tau\theta} = v \frac{\partial u}{\partial \tau}.$$

因此, 支撑函数 h 满足

$$\begin{aligned} h_{\tau\tau} &= \left(\frac{\partial^2 \gamma}{\partial u \partial t} \frac{\partial u}{\partial \tau}, -\vec{N}\right) - k - ch \\ &= kh_{\theta\tau}^2 - k - ch = (h_{\theta\tau}^2 - 1)k - ch \\ &= \frac{h_{\theta\tau}^2 - 1}{h_{\theta\theta} + h} - ch, \end{aligned} \tag{2.1}$$

即

$$hh_{\tau\tau} + ch_{\theta\theta} + h_{\tau\tau}h_{\theta\theta} - h_{\theta\tau}^2 + 1 + ch^2 = 0, \quad (\theta, \tau) \in I \times [0, T]. \tag{2.2}$$

则 (1.1) 可简化为下列方程的 Cauchy 问题:

$$\begin{cases} hh_{\tau\tau} + ch_{\theta\theta} + h_{\tau\tau}h_{\theta\theta} - h_{\theta\tau}^2 + 1 + ch^2 = 0, \\ h(\theta, 0) = g(\theta), \\ h_\tau(\theta, 0) = -\tilde{f}(\theta), \end{cases} \tag{2.3}$$

其中 g 是 γ_0 的支撑函数, $\tilde{f}$ 是初始速度.

对一个定义在 $(\theta, \tau) \in \mathbb{R}^2$ 上的未知函数 $z = z(\theta, \tau)$, 相应的 Monge-Ampère 方程为

$$A + Bz_{\tau\tau} + Cz_{\tau\theta} + Dz_{\theta\theta} + E(z_{\tau\tau}z_{\theta\theta} - z_{\theta\tau}^2) = 0, \tag{2.4}$$

系数 A, B, C, D 和 E 依赖于 τ, θ, z, z_τ 和 z_θ . 我们称 (2.4) 为双曲型 Monge-Ampère 方程, 如果

$$\Delta^2(\tau, \theta, z, z_\tau, z_\theta) \triangleq C^2 - 4BD + 4AE > 0$$

且

$$z_{\theta\theta} + B(\tau, \theta, z, z_\tau, z_\theta) \neq 0.$$

易证, (2.2) 是一个双曲型 Monge-Amère 方程, 其中

$$A = 1 + ch^2, \quad B = h, \quad C = 0, \quad D = ch, \quad E = 1.$$

事实上,

$$\Delta^2(\tau, \theta, h, h_\tau, h_\theta) = C^2 - 4BD + 4A = 0^2 - 4h \cdot ch + 4(1 + ch^2) = 4 > 0$$

且

$$h_{\theta\theta} + B(\tau, \theta, h, h_\tau, h_\theta) = h_{\theta\theta} + h = \frac{1}{k} \neq 0.$$

进而, 假设 $g(\theta)$ 是三次连续可微函数, $\tilde{f}(\theta)$ 是二次连续可微函数, 则初始条件满足

$$\Delta^2(0, \theta, h, -\tilde{f}, g_\theta) = 4 > 0$$

且

$$h_{\theta\theta} + B(0, \theta, h, -\tilde{f}, g_\theta) = g_{\theta\theta} + g = \frac{1}{k_0} \neq 0.$$

由标准的双曲型理论, 我们有局部存在唯一性定理 (即定理 1.1).

3 Riemann 不变量方程组

本节主要是 (2.2) 的约化. 为了简单起见, 我们用 t 代替 τ , 设 $h(\theta, t)$ 是定义在区域 $\tilde{D}$:

$$\tilde{D} = \{(\theta, t) \mid 0 \leq \theta \leq 2\pi, 0 < t < T\}$$

上 (2.2) 的 C^3 -解. 假设

$$h_{\theta\theta} + h \neq 0, \quad \forall (\theta, t) \in \tilde{D}.$$

这个不等式意味着直线 t (常数) 是自由的. 由定义, 取

$$r = \frac{C + \sqrt{\Delta} - 2h_{\theta t}}{2(h_{\theta\theta} + h)} = \frac{1 - h_{\theta t}}{h_{\theta\theta} + h}, \quad s = \frac{C - \sqrt{\Delta} - 2h_{\theta t}}{2(h_{\theta\theta} + h)} = \frac{-1 - h_{\theta t}}{h_{\theta\theta} + h}. \quad (3.1)$$

函数 r 和 s 是 (2.2) 的特征线的斜率, 我们称它们为 (2.2) 的 Riemann 不变量. 设 $p = h_t$, $q = h_\theta$, 则

$$\begin{cases} r_t + sr_\theta = \frac{r-s}{2}(cq - pr), \\ s_t + rs_\theta = \frac{s-r}{2}(ps - cq), \\ h_t + rh_\theta = p + rq, \\ p_t + sp_\theta = s - ch, \\ q_t + rq_\theta = 1 - hr. \end{cases} \quad (3.2)$$

定理 3.1 设 $h(\theta, t)$ 是 (2.3) 的 C^3 -解. 假设 h 满足 (2.3) 和 (3.1), 则函数集 (r, s, h, p, q) 是 (3.2) 定义在 $\tilde{D}$ 上的 C^1 -解, 其中 r 和 s 为 (3.1), $p = h_t$, $q = h_\theta$.

定理 3.2 设 (r, s, S, p, q) 是定义在区域 $\tilde{D}$ 上 (3.2) 的 C^1 -解. 设这个解的初值为

$$\begin{cases} h(0, \theta) = g(\theta), \\ p(0, \theta) = \tilde{f}(\theta), \\ q(0, \theta) = g'(\theta), \\ r(0, \theta) = \frac{1 - \tilde{f}'(\theta)}{g''(\theta) + g(\theta)}, \\ s(0, \theta) = \frac{-1 - \tilde{f}'(\theta)}{g''(\theta) + g(\theta)}. \end{cases}$$

设 $\tilde{D}$ 是这个解的定义域. 假设

$$\Delta(t, \theta, h(t, \theta), h_t(t, \theta), h_\theta(t, \theta)) > 0$$

成立. 则 $h(\theta, t)$ 是定义在区域 $\tilde{D}$ 上 (2.3) 的 C^3 -解, 并且, $h_t = p$, $h_\theta = q$, $h_{\theta\theta} + h \neq 0$ 成立.

若令 $u = (u_1, u_2, u_3, u_4, u_5)^T = (r, s, h, p, q)^T$, $\lambda = \text{diag}_{i=1, \dots, 5} \{\lambda_i\} = \text{diag}\{s, r, r, s, r\}$,

$$\mu = (\mu_1, \dots, \mu_5)^T = \left(\frac{r-s}{2}(cq - pr), \frac{s-r}{2}(ps - cq), p + rq, s - ch, 1 - hr \right)^T,$$

则可将 (3.2) 转化成 Riemann 不变量形式下的拟线性双曲方程组的 Cauchy 问题

$$\begin{cases} \sum_{j=1}^5 \delta_{ij} \left(\frac{\partial u_j}{\partial t} + \lambda_i \frac{\partial u_j}{\partial \theta} \right) = \mu_i, & i = 1, \dots, 5, \\ u_i(0, \theta) = \phi_i(\theta), & i = 1, \dots, 5, \end{cases} \quad (3.3)$$

其中

$$\phi = (\phi_1, \dots, \phi_5)^T = \left(\frac{1 - \tilde{f}'(\theta)}{g''(\theta) + g(\theta)}, \frac{-1 - \tilde{f}'(\theta)}{g''(\theta) + g(\theta)}, g(\theta), \tilde{f}(\theta), g'(\theta) \right)^T.$$

首先, 假设 $\phi = (\phi_1, \dots, \phi_5)^T$ 是 R 上有界的, 且 $\phi \in C^1(R)$. 选取适当的正常数 Ω , 使其满足

$$\|\phi\| < \Omega,$$

其中模 $\|\cdot\|$ 为 C^0 空间的模. 在此处及以下的估计中, 对任何向量 $u = (u_i)$, 记

$$|u| = \max_i |u_i|,$$

区域 R 上的连续的向量函数 $u = u(t, \theta) = (u_i(t, \theta))$ 的模 $\|u\|$ 定义为

$$\|u\| = \sup_{(t, \theta) \in R} |u(t, \theta)|,$$

而对连续可微的向量函数 $u(t, \theta)$ 还定义

$$\|u\|_1 = \|u\| + \left\| \frac{\partial u}{\partial t} \right\| + \left\| \frac{\partial u}{\partial \theta} \right\|.$$

同时, 对任意有限的函数集合 Γ , 也可类似的定义它的模 $\|\Gamma\|$ 和 $\|\Gamma\|_1$.

记 $R(\delta_0) = \{(t, \theta) \mid t \in [0, \delta_0], \theta \in R^1\}$, 并将其扩展区域定义为

$$E(\delta_0) = \{(t, \theta, u) \mid (t, \theta) \in R(\delta_0), |u| \leq \Omega\}.$$

进一步假设, $\lambda_i, \frac{\partial \lambda_i}{\partial \theta}, \frac{\partial \lambda_i}{\partial u_j}, \mu_i, \frac{\partial \mu_i}{\partial \theta}$ 和 $\frac{\partial \mu_i}{\partial u_j}$ 是 $E(\delta_0)$ 上的 C^0 函数, 那么, 由标准的双曲型理论, 存在常数 $0 < \delta_* \leq \delta_0$, 使得 (3.3) 的 Cauchy 问题在 $R(\delta_*)$ 上有唯一的 C^1 解 $u(t, \theta)$.

4 定理 1.2 的证明

本节将利用拟线性双曲方程组的 Cauchy 问题的局部解的构造方法来考虑局部解的生命跨度 (解的存在区间的下界).

首先, 考虑将拟线性双曲方程组 (3.3) 线性化. 选取适当的正常数 Ω_1 , 使得

$$\Omega_1 > \|\phi\| + (1 + \Omega)\|\dot{\phi}\| + \Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1,$$

这里对 Ω_1 的要求来源于第二先验估计. 引入下列函数集合

$$\Sigma(\delta | \Omega, \Omega_1) = \{\nu(t, \theta) | \nu = (\nu_1, \dots, \nu_5)^T \in C^1[R(\delta)], \|\nu\| \leq \Omega, \|\nu\|_1 \leq \Omega_1\}.$$

对于任意 $\nu \in \Sigma(\delta | \Omega, \Omega_1)$, 由线性方程组

$$\sum_{j=1}^5 \delta_{lj} \left(\frac{\partial u_j}{\partial t} + \tilde{\lambda}_l(t, \theta) \frac{\partial u_j}{\partial \theta} \right) = \tilde{\mu}_l(t, \theta), \quad l = 1, \dots, 5, \quad (4.1)$$

及初始条件 $u_l(0, \theta) = \phi_l(\theta)$, $\theta \in R^1$, $l = 1, \dots, 5$, 其中 $\tilde{\lambda}_l(t, \theta) = \lambda_l(t, \theta, \nu)$, 即 $\tilde{\lambda} = (\nu_2, \nu_1, \nu_1, \nu_2, \nu_1)^T$, $\tilde{\mu}_l(t, \theta) = \mu_l(t, \theta, \nu)$, 即

$$\tilde{\mu} = \mu(t, \theta, \nu) = \left(\frac{\nu_1 - \nu_2}{2}(c\nu_5 - \nu_4\nu_1), \frac{\nu_2 - \nu_1}{2}(\nu_4\nu_2 - c\nu_5), \nu_4 + \nu_1\nu_5, \nu_2 - c\nu_3, 1 - \nu_3\nu_1 \right)^T.$$

令函数集合

$$\Gamma^* = \left\{ \lambda_l, \frac{\partial \lambda_l}{\partial \theta}, \frac{\partial \lambda_l}{\partial u_j}, \mu_l, \frac{\partial \mu_l}{\partial \theta}, \frac{\partial \mu_l}{\partial u_j}, l, j = 1, \dots, 5 \right\},$$

对该集合作 C^0 模估计,

$$\begin{aligned} \max_{l=1, \dots, 5} |\lambda_l(t, \theta, \nu)| &= \max\{|\nu_1|, |\nu_2|\} \leq \Omega, \\ \max_{l=1, \dots, 5} \left| \frac{\partial \lambda_l}{\partial \theta}(t, \theta, \nu) \right| &= 0, \quad \max_{l, j=1, \dots, 5} \left| \frac{\partial \lambda_l}{\partial u_j}(t, \theta, \nu) \right| = 1, \\ \max_{l=1, \dots, 5} |\mu_l(t, \theta, \nu)| &\leq \max\{\Omega(|c|\Omega + \Omega^2), \Omega(|c|\Omega + \Omega^2), \Omega + \Omega^2, (1 + |c|)\Omega, 1 + \Omega^2\} \\ &\leq \Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1, \\ \max_{l=1, \dots, 5} \left| \frac{\partial \mu_l}{\partial \theta}(t, \theta, \nu) \right| &= 0, \\ \left(\frac{\partial \mu_l}{\partial u_j}(t, \theta, \nu) \right) &= \begin{pmatrix} \frac{1}{2}[\nu_4(\nu_2 - 2\nu_1) + c\nu_5] & -\frac{1}{2}[\nu_4\nu_2 - c\nu_5] & \nu_5 & 0 & -\nu_3 \\ -\frac{1}{2}[c\nu_5 - \nu_4\nu_1] & \frac{1}{2}[\nu_4(2\nu_2 - \nu_1) - c\nu_5] & 0 & 1 & 0 \\ 0 & 0 & 0 & -c & -\nu_1 \\ -\frac{1}{2}\nu_1(\nu_1 - \nu_2) & \frac{1}{2}\nu_2(\nu_2 - \nu_1) & 1 & 0 & 0 \\ \frac{c}{2}(\nu_1 - \nu_2) & \frac{c}{2}(\nu_1 - \nu_2) & \nu_1 & 0 & 0 \end{pmatrix}, \\ \max_{l, j=1, \dots, 5} \left| \frac{\partial \mu_l}{\partial u_j}(t, \theta, \nu) \right| &\leq \max \left\{ \frac{1}{2}(3\Omega^2 + |c|\Omega), |c|\Omega, \Omega, |c|, 1 \right\} \\ &\leq \frac{3}{2}\Omega^2 + \max\{|c|, 1\}(\Omega + 1), \end{aligned}$$

综上所述, Γ^* 的 C^0 模为

$$\|\Gamma^*\| = \max \left\{ \Omega, 0, 1, \Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1, \frac{3}{2}\Omega^2 + \max\{|c|, 1\}(\Omega + 1) \right\}.$$

令

$$\Gamma_2[\nu] = \left\{ \tilde{\lambda}_l(t, \theta), \frac{\partial \tilde{\lambda}_l}{\partial \theta}(t, \theta), \tilde{\mu}_l(t, \theta), \frac{\partial \tilde{\mu}_l}{\partial \theta}(t, \theta) \right\},$$

则有

$$\frac{\partial \tilde{\lambda}_l}{\partial \theta}(t, \theta) = \frac{\partial \lambda_l}{\partial \theta}(t, \theta, \nu) + \sum_{j=1}^5 \frac{\partial \lambda_l}{\partial u_j}(t, \theta, \nu) \frac{\partial \nu_j}{\partial \theta}, \quad l = 1, \dots, 5, \quad (4.2)$$

$$\frac{\partial \tilde{\mu}_l}{\partial \theta}(t, \theta) = \frac{\partial \mu_l}{\partial \theta}(t, \theta, \nu) + \sum_{j=1}^5 \frac{\partial \mu_l}{\partial u_j}(t, \theta, \nu) \frac{\partial \nu_j}{\partial \theta}, \quad l = 1, \dots, 5, \quad (4.3)$$

$$\frac{\partial \tilde{\lambda}_l}{\partial \theta} \leq \Omega_1,$$

$$\frac{\partial \tilde{\mu}_l}{\partial \theta} \leq \Omega_1 \max\{3\Omega^2 + 2|c|\Omega, 2\Omega + 1, 1 + |c|\} \leq (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1.$$

因此, $\Gamma_2[\nu]$ 的 C^0 模为

$$\Gamma_2[\nu] = \max\{\Omega, \Omega_1, \Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1, (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\}.$$

下面将给出拟线性双曲型方程组 Cauchy 问题解的某些先验估计. 首先, 由第一相关积分,

$$u_k(t, \theta) = u_k^0(t, \theta) + \int_0^t \mu_k(\tau, \omega_k(\tau; t, \theta)) d\tau, \quad k = 1, \dots, 5, \quad (4.4)$$

初始条件 $u_k(0, \theta) = \phi_k(\omega_k(0; t, \theta))$, $k = 1, \dots, 5$. 在 (4.4) 的右端及在本节中第 l 个积分的被积函数的自变量均取为 $(\tau, \xi) = (\tau, \omega_l(\tau; t, \theta))$, 此表示所作的积分是沿第 l 族特征曲线进行的; 而记号 $\frac{d}{d\tau}$ 表示沿第 l 族特征线的微商, 即

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \lambda_l \frac{\partial}{\partial \xi}.$$

由

$$\max_{k=1, \dots, 5} |u_k^0(t, \theta)| \leq \|\phi\|$$

和

$$\max_{l=1, \dots, 5} |\mu_l(t, \theta, \nu)| \leq \Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1,$$

当 $0 \leq t \leq \delta$ 时,

$$\|u\| \leq \|\phi\| + (\Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1)\delta. \quad (4.5)$$

现在对 Cauchy 问题的解的一阶导数的上界建立第二估计式. 若记函数集合

$$p = (p_i(t, \theta)) = \left(\frac{\partial u_i(t, \theta)}{\partial t} \right), \quad w = (w_i(t, \theta)) = \left(\frac{\partial u_i(t, \theta)}{\partial \theta} \right), \quad i = 1, \dots, 5, \quad (4.6)$$

$$\bar{\mu}_l(t, \theta) = \frac{\partial \mu_l(t, \theta)}{\partial \theta} - \frac{\lambda_l}{\partial \theta} w_l, \quad l = 1, \dots, 5,$$

由第二相关积分

$$w_k(t, \theta) = w_k^0(t, \theta) + \int_0^t \bar{\mu}_k(\tau, \omega_k(\tau; t, \theta)) d\tau, \quad k = 1, \dots, 5, \quad (4.7)$$

其中

$$\begin{aligned} w_k^0(t, \theta) &= \dot{\phi}_k(\omega_k(0; t, \theta)), \quad k = 1, \dots, 5, \\ p_k(t, \theta) &= p_k^0(t, \theta) - \lambda_k \int_0^t \bar{\mu}_k(\tau, \omega_k(\tau; t, \theta)) d\tau, \quad k = 1, \dots, 5, \\ p_k^0(t, \theta) &= \mu_k(t, \theta) - \lambda_l(t, \theta) \dot{\phi}_k(\omega_k(0; t, \theta)), \quad k = 1, \dots, 5. \end{aligned} \quad (4.8)$$

则在 $R(\delta)$ 上,

$$\begin{aligned} \max_{k=1, \dots, 5} |w_k^0(t, \theta)| &\leq \|\dot{\phi}\|, \\ \max_{k=1, \dots, 5} |p_k^0(t, \theta)| &\leq \Omega \|\dot{\phi}\| + \Omega^3 + \max\{|c|, 1\} \Omega^2 + (1 + |c|) \Omega + 1. \end{aligned}$$

令

$$w(\tau) = \sup_{k=1, \dots, 5} \{|w_k(t, \theta)|, (t, \theta) \in R(\tau)\}, \quad p(\tau) = \sup_{k=1, \dots, 5} \{|p_k(t, \theta)|, (t, \theta) \in R(\tau)\},$$

则显然有

$$\|w\| = \sup_{0 \leq \tau \leq \delta} w(\tau) = w(\delta), \quad \|p\| = \sup_{0 \leq \tau \leq \delta} p(\tau) = p(\delta).$$

由积分关系式, 当 $0 \leq t \leq \delta$ 时,

$$w(t) \leq \|\dot{\phi}\| + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\delta + \Omega_1 \int_0^t w(\tau) d\tau.$$

由 Gronwall 不等式,

$$w(t) \leq [\|\dot{\phi}\| + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\delta] e^{\Omega_1 t},$$

从而有

$$\begin{aligned} \|w\| &\leq [\|\dot{\phi}\| + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\delta] e^{\Omega_1 \delta}, \\ p(t) &\leq \Omega \|\dot{\phi}\| + \Omega^3 + \max\{|c|, 1\} \Omega^2 + (1 + |c|) \Omega + 1 \\ &\quad + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega\Omega_1\delta + \Omega\Omega_1 \int_0^t w(\tau) d\tau. \end{aligned} \quad (4.9)$$

将 $w(t)$ 满足的不等式代入上式, 得

$$p(t) \leq \Omega^3 + \max\{|c|, 1\} \Omega^2 + (1 + |c|) \Omega + 1 + \Omega [\|\dot{\phi}\| + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\delta] e^{\Omega_1 t},$$

从而有

$$\|p\| \leq \Omega^3 + \max\{|c|, 1\} \Omega^2 + (1 + |c|) \Omega + 1 + \Omega [\|\dot{\phi}\| + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\delta] e^{\Omega_1 \delta}. \quad (4.10)$$

于是, 由 $\|u\|_1 = \|u\| + \|p\| + \|w\|$ 可知,

$$\begin{aligned} \|u\|_1 &\leq \|\phi\| + [\Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1](1 + \delta) \\ &\quad + (\Omega + 1)[\|\dot{\phi}\| + (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\delta]e^{\Omega_1\delta}, \end{aligned} \quad (4.11)$$

假设 $\delta \leq \frac{1}{\Omega_1}$, 则

$$e^{\Omega_1\delta} \leq e^1 \leq 3.$$

由 Taylor 展开式,

$$e^{\Omega_1\delta} < 1 + \Omega_1\delta e^{\Omega_1\delta} < 1 + 3\Omega_1\delta,$$

将上式代入 (4.11), 并简化 δ 的二阶项,

$$\begin{aligned} \|u\|_1 &\leq \|\phi\| + \Omega^3 + \max\{|c|, 1\}\Omega^2 + |c|\Omega + (\|\dot{\phi}\| + 1)(\Omega + 1) \\ &\quad + (\Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1)\delta \\ &\quad + (\Omega + 1)\Omega_1(12\Omega^2 + 8\max\{|c|, 1\}\Omega + 4|c| + 4 + 3\|\dot{\phi}\|)\delta. \end{aligned} \quad (4.12)$$

以下将对解的一阶偏导数的连续性模进行估计, 称为 Cauchy 问题的第三估计式. 为此我们先给出连续性模的定义. 设 $\psi(t, \theta)$ 是定义在区域 R 上的函数, 我们称非负值函数

$$\rho(\eta) = \rho(\eta | \psi) = \sup_{|t' - t''| \leq \eta, |\theta' - \theta''| \leq \eta} \{|\psi(t', \theta') - \psi(t'', \theta'')|, (t', \theta'), (t'', \theta'') \in (0 \leq \eta < \infty)\} \quad (4.13)$$

为 $\psi(t, \theta)$ 在 R 上的连续性模. 类似地, 可分别定义向量函数 $\psi = (\psi_i(t, \theta))$ ($i = 1, \dots, n$) 或一般的有限个函数的集合 $\Gamma = \{\psi_i\}$ 的连续性模为

$$\rho(\eta | \psi) = \max_{i=1, \dots, n} \rho(\eta | \psi_i), \quad (4.14)$$

$$\rho(\eta | \Gamma) = \max_{\psi_i \in \Gamma} \rho(\eta | \psi_i). \quad (4.15)$$

由 $\theta_i = \omega_l(\tau; t, \theta)$ 满足常微分方程

$$\frac{d\xi}{d\tau} = \lambda_l(\tau, \xi) \quad (4.16)$$

和初始条件

$$\tau = t : \omega_l(t; t, \theta) = \theta. \quad (4.17)$$

易知, 有

$$\begin{cases} \frac{\partial \omega_l}{\partial \theta}(\tau; t, \theta) = \exp \left\{ \int_t^\tau \frac{\partial \lambda_l}{\partial \xi}(\tau_1, \omega_l(\tau_1; t, \theta)) d\tau_1 \right\}, \\ \frac{\partial \omega_l}{\partial t}(\tau; t, \theta) = -\lambda_l \exp \left\{ \int_t^\tau \frac{\partial \lambda_l}{\partial \xi}(\tau_1, \omega_l(\tau_1; t, \theta)) d\tau_1 \right\}, \end{cases} \quad (4.18)$$

因此,

$$\left| \frac{\partial \omega_l}{\partial \theta} \right| \leq \exp \left(\left\| \frac{\partial \lambda_l}{\partial \theta} \right\| t \right) \leq 1 + \left\| \frac{\partial \lambda_l}{\partial \theta} \right\| \delta \exp \left(\left\| \frac{\partial \lambda_l}{\partial \theta} \right\| \delta \right), \quad l = 1, \dots, 5, \quad (4.19)$$

$$\left| \frac{\partial \omega_l}{\partial t} \right| \leq \|\lambda_l\| \exp \left(\left\| \frac{\partial \lambda_l}{\partial \theta} \right\| t \right) \leq \|\lambda_l\| \left[1 + \left\| \frac{\partial \lambda_l}{\partial \theta} \right\| \delta \exp \left(\left\| \frac{\partial \lambda_l}{\partial \theta} \right\| \delta \right) \right], \quad l = 1, \dots, 5. \quad (4.20)$$

由 $\delta \leq \frac{1}{\Omega_1}$,

$$\left\| \frac{\partial \omega}{\partial \theta} \right\| \leq 1 + 3\Omega_1 \delta, \quad \left\| \frac{\partial \omega}{\partial t} \right\| \leq (1 + 3\Omega_1 \delta) \Omega. \quad (4.21)$$

$$\rho(\eta | w_k^0(t, \theta)) = \rho(\eta | \dot{\phi}(\omega_k(0; t, \theta))) \leq (1 + \Omega)(1 + 3\Omega_1 \delta) \rho(\eta | \dot{\phi}_k), \quad (4.22)$$

$$\begin{aligned} \rho(\eta | p_k^0(t, \theta)) &= \rho(\eta | \mu_k - \lambda_k \dot{\phi}(\omega_k(0; t, \theta))) \\ &\leq \rho(\eta | \mu_k) + \|\lambda_k\| \rho(\eta | \dot{\phi}(\omega_k(0; t, \theta))) + \|\dot{\phi}_k\| \rho(\eta | \lambda_k). \end{aligned} \quad (4.23)$$

记 $\chi(\eta) = \max\{\rho(\eta | \mu_k), \rho(\eta | \lambda_k)\}$, 则可得

$$\rho(\eta | p_k^0(t, \theta)) \leq \Omega(1 + \Omega)(1 + 3\Omega_1 \delta) \rho(\eta | \dot{\phi}_k) + (1 + \|\dot{\phi}\|) \chi(\eta). \quad (4.24)$$

综上所述,

$$\rho(\eta | w^0) \leq (1 + \Omega)(1 + 3\Omega_1 \delta) \rho(\eta | \dot{\phi}), \quad (4.25)$$

$$\rho(\eta | p^0) \leq \Omega(1 + \Omega)(1 + 3\Omega_1 \delta) \rho(\eta | \dot{\phi}) + (1 + \|\dot{\phi}\|) \chi(\eta). \quad (4.26)$$

两式相加,

$$\rho(\eta | w^0) + \rho(\eta | p^0) \leq (1 + \Omega)^2 (1 + 3\Omega_1 \delta) \rho(\eta | \dot{\phi}) + (1 + \|\dot{\phi}\|) \chi(\eta), \quad (4.27)$$

$$\rho(\tau, \eta | w) = \sup_{k=1, \dots, 5, (t', \theta'), (t'', \theta'') \in R(\tau) | t' - t'' \leq \eta, |\theta' - \theta''| \leq \eta} |w_k(t', \theta') - w_k(t'', \theta'')|, \quad (4.28)$$

$$\rho(\tau, \eta | p) = \sup_{k=1, \dots, 5, (t', \theta'), (t'', \theta'') \in R(\tau) | t' - t'' \leq \eta, |\theta' - \theta''| \leq \eta} |p_k(t', \theta') - p_k(t'', \theta'')|. \quad (4.29)$$

由第二相关积分, 对 $\rho(\tau, \eta | w)$ 和 $\rho(\tau, \eta | p)$ 进行估计. 当 $\bar{t} \leq \delta$ 时,

$$\rho(\bar{t}, \eta | \bar{\mu}_k(\tau, \omega_k(\tau; t, \theta))) \leq 4(1 + \Omega) \rho(\tau, \eta | \bar{\mu}_k(\tau, \xi)), \quad (4.30)$$

$$\begin{aligned} \rho(\tau, \eta | \bar{\mu}_k(\tau, \xi)) &= \rho\left(\tau, \eta \left| \frac{\partial \mu_k(\tau, \xi)}{\partial \theta} - \frac{\partial \lambda_k(\tau, \xi)}{\partial \theta} w_k(\tau, \xi) \right.\right) \\ &\leq \rho\left(\tau, \eta \left| \frac{\partial \mu_k}{\partial \theta} \right.\right) + \left\| \frac{\partial \lambda_k}{\partial \theta} \right\| \rho(\tau, \eta | w_k(\tau, \xi)) + \|w_k\| \rho\left(\tau, \eta \left| \frac{\partial \lambda_k}{\partial \theta}(\tau, \xi) \right.\right) \\ &\leq \Omega_1 \rho(\tau, \eta | w) + (1 + \|w\|) \chi_1(\eta) \\ &\leq \Omega_1 \rho(\tau, \eta | w) + (4\|\dot{\phi}\| + 12\Omega^2 + 8 \max\{|c|, 1\} \Omega + 5 + 4|c|) \chi_1(\eta), \end{aligned} \quad (4.31)$$

其中

$$\begin{aligned} \chi_1(\eta) &= \max_{k=1, \dots, 5} \left\{ \rho\left(\tau, \eta \left| \frac{\partial \mu_k}{\partial \theta} \right.\right), \rho\left(\tau, \eta \left| \frac{\partial \lambda_k}{\partial \theta} \right.\right) \right\}, \\ \rho(\bar{t}, \eta | \bar{\mu}_k(\tau, \omega_k(\tau; t, \theta))) &\leq 4(1 + \Omega) [\Omega_1 \rho(\tau, \eta | w) + (4\|\dot{\phi}\| + 12\Omega^2 \\ &\quad + 8 \max\{|c|, 1\} \Omega + 5 + 4|c|) \chi_1(\eta)]. \end{aligned} \quad (4.32)$$

由 $\bar{\mu}_k = \frac{\partial \tilde{\mu}_k}{\partial \theta} - \frac{\partial \tilde{\lambda}_k}{\partial \theta} w_k$, $k = 1, \dots, 5$ 得

$$\|\bar{\mu}_k\| \leq \left\| \frac{\partial \tilde{\mu}_k}{\partial \theta} \right\| + \left\| \frac{\partial \tilde{\lambda}_k}{\partial \theta} \right\| \cdot \|w_k\| \leq (15\Omega^2 + 10 \max\{|c|, 1\} \Omega + 5|c| + 5 + 4\|\dot{\phi}\|) \Omega_1, \quad (4.33)$$

即

$$\|\bar{\mu}\| \leq (15\Omega^2 + 10 \max\{|c|, 1\}\Omega + 5|c| + 5 + 4\|\dot{\phi}\|)\Omega_1. \quad (4.34)$$

由连续模的性质

$$\rho\left(\eta \mid \int_{a(t)}^{b(t)} \psi(\tau, t) d\tau\right) \leq \|\psi\|[\rho(\eta \mid a) + \rho(\eta \mid b)] + \sup_t \int_{a(t)}^{b(t)} \rho(\eta \mid \psi(\tau, t)) d\tau \quad (4.35)$$

可得

$$\begin{aligned} & \rho\left(\bar{t}, \eta \mid \int_0^t \bar{\mu}_k(\tau, \omega_k(\tau; t, \theta)) d\tau\right) \\ & \leq \|\bar{\mu}_k\| \eta + \int_0^{\bar{t}} \rho(\tau_1, \eta \mid \bar{\mu}_k(\tau, \omega_k(\tau; t, \theta))) d\tau_1 \\ & \leq (15\Omega^2 + 10 \max\{|c|, 1\}\Omega + 5|c| + 5 + 4\|\dot{\phi}\|)\Omega_1 \eta + 4(1 + \Omega)\Omega_1 \int_0^{\bar{t}} \rho(\tau, \eta \mid w) d\tau \\ & \quad + 4(1 + \Omega)(4\|\dot{\phi}\| + 12\Omega^2 + 8 \max\{|c|, 1\}\Omega + 5 + 4|c|)\delta\chi_1(\eta). \end{aligned} \quad (4.36)$$

因此,

$$\begin{aligned} \rho(t, \eta \mid p_k) + \rho(t, \eta \mid w_k) & \leq \rho(t, \eta \mid p_k^0) + \rho(t, \eta \mid w_k^0) + (1 + \|\lambda\|)\rho\left(t, \eta \mid \int_0^t \bar{\mu}_k d\tau\right) + \left\|\int_0^t \bar{\mu}_k\right\| \chi(\eta) \\ & \leq [1 + \|\dot{\phi}\| + (1 + \Omega_1 + \Omega\Omega_1)(15\Omega^2 + 10 \max\{|c|, 1\}\Omega \\ & \quad + 5|c| + 5 + 4\|\dot{\phi}\|)]\chi(\eta) + 4(1 + \Omega)^2(4\|\dot{\phi}\| + 12\Omega^2 \\ & \quad + 8 \max\{|c|, 1\}\Omega + 5 + 4|c|)\chi_1(\eta)\delta\chi_1(\eta) + 4(1 + \Omega)^2\rho(\eta \mid \dot{\phi}) \\ & \quad + 4(1 + \Omega)^2\Omega_1 \left[\int_0^t \rho(\tau, \eta \mid w) d\tau + \int_0^t \rho(\tau, \eta \mid p) d\tau \right]. \end{aligned} \quad (4.37)$$

从而, 由 Gronwall 不等式, 可得

$$\begin{aligned} & \rho(t, \eta \mid p) + \rho(t, \eta \mid w) \\ & \leq e^{4(1+\Omega)^2} [1 + \|\dot{\phi}\| + (1 + \Omega_1 + \Omega\Omega_1)(15\Omega^2 + 10 \max\{|c|, 1\}\Omega + 5|c| + 5 + 4\|\dot{\phi}\|)]\chi(\eta) \\ & \quad \times e^{4(1+\Omega)^2} [4(1 + \Omega)^2\rho(\eta \mid \dot{\phi}) + 4(1 + \Omega)^2(4\|\dot{\phi}\| + 12\Omega^2 + 8 \max\{|c|, 1\}\Omega + 5 + 4|c|)\delta\chi_1(\eta)]. \end{aligned} \quad (4.38)$$

一致有界性的构造由线性双曲方程组的 Cauchy 问题知, 在区域 $R(\delta)$ 上唯一的确定一连续可微解 $u = (u_i(t, \theta))$. 记 $u = Tv$. 显然, 算子 T 的不动点就是所考察的 Cauchy 问题的解, 因此, 我们只要证明当 δ 足够小时, 算子 T 有唯一的不动点. 首先, 我们证明可以找到足够小的正数 $\delta_1 = \delta_1(\|r\|, \|\phi\|_1)$, 当 $\delta \leq \delta_1$ 时, 算子 T 确定了从集合 $\Sigma(\delta \mid \Omega, \Omega_1)$ 到自身的变换. 在 $R(\delta)$ 上, 对于 $u = Tv$ 有如下的估计式:

$$\|u\| \leq \|\phi\| + (\Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1)\delta. \quad (4.39)$$

取 $\Omega = 2\|\phi\|$, 则

$$\delta = \frac{\|\phi\|}{8\|\phi\|^3 + 4 \max\{|c|, 1\}\|\phi\|^2 + 2(1 + |c|)\|\phi\| + 1}, \quad (4.40)$$

有 $\|u\| \leq \Omega$,

$$\|u\|_1 \leq \|\phi\| + \Omega^3 + \max\{|c|, 1\}\Omega^2 + |c|\Omega + (\|\dot{\phi}\| + 1)(\Omega + 1)$$

$$\begin{aligned}
& + (\Omega^3 + \max\{|c|, 1\}\Omega^2 + (1 + |c|)\Omega + 1)\delta \\
& + (\Omega + 1)\Omega_1(12\Omega^2 + 8\max\{|c|, 1\}\Omega + 4|c| + 4 + 3\|\dot{\phi}\|)\delta.
\end{aligned} \quad (4.41)$$

取

$$\begin{aligned}
\Omega_1 &= 2\|\phi\| + 2\Omega^3 + 2\max\{|c|, 1\}\Omega^2 + 2|c|\Omega + 2(\|\dot{\phi}\| + 1)(\Omega + 1) \\
&= (4\|\phi\| + 2)\|\dot{\phi}\| + (16\|\phi\|^3 + 8\max\{|c|, 1\}\|\phi\|^2 + 4|c|\|\phi\| + 6\|\phi\| + 2).
\end{aligned} \quad (4.42)$$

记

$$\begin{aligned}
A_0(\|\phi\|) &= 8\|\phi\|^3 + 4\max\{|c|, 1\}\|\phi\|^2 + 2|c|\|\phi\| + 3\|\phi\| + 1, \quad A_1(\|\phi\|) = 2\|\phi\| + 1, \\
A_2(\|\phi\|) &= A_0 - \|\phi\|, \quad A_3(\|\phi\|) = 48\|\phi\|^2 + 16\max\{|c|, 1\}\|\phi\| + 4|c| + 4, \\
B_0(\|\phi\|) &= 2A_0A_1A_3 + A_2, \quad B_1(\|\phi\|) = 6A_0A_1 + 2A_1^2A_3, \quad B_2(\|\phi\|) = 6A_1^2,
\end{aligned}$$

则可取

$$\delta = \frac{A_0 + A_1\|\dot{\phi}\|}{B_0 + B_1\|\dot{\phi}\| + B_2\|\dot{\phi}\|^2}. \quad (4.43)$$

于是, 取

$$\delta_1 = \min \left\{ \frac{\|\phi\|}{A_0 - \|\phi\|}, \frac{A_0 + A_1\|\dot{\phi}\|}{B_0 + B_1\|\dot{\phi}\| + B_2\|\dot{\phi}\|^2} \right\}, \quad (4.44)$$

当 $\delta \leq \delta_1$ 时, 算子 T 是一个从函数集合 $\Sigma(\delta | \Omega, \Omega_1)$ 映为自身的算子.

现在考虑 δ_2 的构造, 使得算子 T 是等度连续的.

引理 4.1 存在正常数 $\delta_2 = \delta_2(\|\phi\|, \|\dot{\phi}\|) \leq \delta_1$ 和非负函数 $\Omega_2(\eta)$ ($0 < \eta < \infty$), 使得

(1) $\lim_{\eta \rightarrow 0} \Omega_2(\eta) = 0$;

(2) 对任意的 $\delta \leq \delta_2$, 如果有 $v \in \Sigma(\delta | \Omega, \Omega_1)$, 且满足

$$\rho\left(\eta \mid \frac{\partial v}{\partial t}\right) + \rho\left(\eta \mid \frac{\partial v}{\partial \theta}\right) \leq \Omega_2(\eta),$$

则对于 $u = Tv$, 有 $\rho(\eta | p) + \rho(\eta | w) \leq \Omega_2(\eta)$.

证明 令

$$\tilde{\chi}(\eta) = \max_{l=1, \dots, 5} \{\rho(\eta | \tilde{\lambda}_l), \rho(\eta | \tilde{\mu}_l), \eta\},$$

根据连续模的定义和 Ω_1 的构造, 则有

$$\tilde{\chi}(\eta) \leq \max\{\Omega_1\eta, (3\Omega^2 + M + 2\Omega + 1)\Omega_1\eta, \eta\} = (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\Omega_1\eta, \quad (4.45)$$

$$\begin{aligned}
\rho(t, \eta | p) + \rho(t, \eta | w) &\leq e^{4(1+\Omega)^2} [1 + \|\dot{\phi}\| + (1 + \Omega_1 + \Omega\Omega_1)(15\Omega^2 + 10\max\{|c|, 1\}\Omega \\
&\quad + 5|c| + 5 + 4\|\dot{\phi}\|)] \chi(\eta) e^{4(1+\Omega)^2} [4(1 + \Omega)^2 \rho(\eta | \dot{\phi}) + 4(1 + \Omega)^2 \\
&\quad \times (4\|\dot{\phi}\| + 12\Omega^2 + 8\max\{|c|, 1\}\Omega + 5 + 4|c|) \delta \chi_1(\eta)],
\end{aligned} \quad (4.46)$$

其中

$$\chi_1(\eta) = \max_{k=1, \dots, 5} \left\{ \rho\left(\tau, \eta \mid \frac{\partial \tilde{\mu}_k}{\partial \theta}\right), \rho\left(\tau, \eta \mid \frac{\partial \tilde{\lambda}_k}{\partial \theta}\right) \right\}.$$

又由于

$$\begin{aligned}\rho\left(\tau, \eta \mid \frac{\partial \tilde{\lambda}_k}{\partial \theta}\right) &\leq \max \left\{ \rho\left(\tau, \eta \mid \frac{\partial \nu_1}{\partial \theta}\right), \rho\left(\tau, \eta \mid \frac{\partial \nu_2}{\partial \theta}\right) \right\} \leq \rho\left(\tau, \eta \mid \frac{\partial \nu}{\partial \theta}\right), \\ \rho\left(\tau, \eta \mid \frac{\partial \tilde{\mu}_k}{\partial \theta}\right) &\leq \max \left\{ (3\Omega^2 + 2|c|\Omega)\rho\left(\tau, \eta \mid \frac{\partial \nu}{\partial \theta}\right), (2\Omega + 1)\rho\left(\tau, \eta \mid \frac{\partial \nu}{\partial \theta}\right), (1 + |c|)\rho\left(\tau, \eta \mid \frac{\partial \nu}{\partial \theta}\right) \right\} \\ &\leq (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\rho\left(\tau, \eta \mid \frac{\partial \nu}{\partial \theta}\right),\end{aligned}$$

因此,

$$\chi_1(\eta) \leq (3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|)\rho\left(\eta \mid \frac{\partial \nu}{\partial \theta}\right).$$

取

$$\begin{aligned}\Omega_2(\eta) &= e^{4(1+\Omega)^2} \{4(1 + \Omega^2)\rho(\eta \mid \dot{\phi})(3\Omega^2 + 2\max\{|c|, 1\}\Omega + 1 + |c|) \\ &\quad \times [1 + \|\dot{\phi}\| + (1 + \Omega_1 + \Omega\Omega_1)(15\Omega^2 + 10\max\{|c|, 1\}\Omega + 5|c| + 5 + 4\|\dot{\phi}\|)]\Omega_1\eta\},\end{aligned}\quad (4.47)$$

则可取

$$\delta_2 = \frac{1}{4(1 + \Omega)^2(4\|\dot{\phi}\| + 12\Omega^2 + 8\max\{|c|, 1\}\Omega + 4|c| + 5)(3\Omega^2 + 2\max\{|c|, 1\}\Omega + |c| + 1)}.\quad (4.48)$$

令

$$\begin{aligned}C_0(\|\phi\|) &= 4(1 + 2\|\phi\|)^2(48\|\phi\|^2 + 16\max\{|c|, 1\}\|\phi\| + 4|c| + 5) \\ &\quad \times (12\|\phi\|^2 + 4\max\{|c|, 1\}\|\phi\| + |c| + 1), \\ C_1(\|\phi\|) &= 16(1 + 2\|\phi\|)^2(12\|\phi\|^2 + 4\max\{|c|, 1\}\|\phi\| + |c| + 1),\end{aligned}$$

则

$$\delta_2 = \frac{1}{C_0 + C_1\|\dot{\phi}\|}.\quad (4.49)$$

那么, 当 $0 \leq \delta\delta_2$ 时, 定理 4.1(1) 和 4.1(2) 成立. 于是取 $\delta_3 = \min\{\delta_1, \delta_2\}$, 则可以引入函数集合

$$\Sigma(\delta_3) = \Sigma(\delta_3 \mid \Omega, \Omega_1, \Omega_2(\eta)) = \left\{ \nu(t, \theta) \mid \nu \in \Sigma(\delta_3 \mid \Omega, \Omega_1), \rho\left(\eta \mid \frac{\partial \nu}{\partial \theta}\right) \leq \Omega_2(\eta) \right\},\quad (4.50)$$

则算子 T 是一个将 $\Sigma(\delta_3)$ 映到自身的算子. $\Sigma(\delta_3)$ 在 $C^1[R(\delta)]$ 上是紧的. (首先, 在 C^1 模下, $\Sigma(\delta_3)$ 集合一致有界且等度连续, 所以是准紧的. 又假设 $\lim_{n \rightarrow \infty} \nu_n = \nu$, 则

$$\rho(\eta \mid \nu) = \lim_{n \rightarrow \infty} \rho(\eta \mid \nu_n) \leq \Omega_2(\eta), \quad \|\nu\| \leq \Omega, \quad \|\nu\|_1 \leq \Omega_1,$$

所以, 该集合在 $C^1[R(\delta)]$ 中是闭的, 即为紧的, 从而, $\Sigma(\delta_3)$ 是完备的.) $\Sigma(\delta_3)$ 在 C^0 模下是闭的 (因为 $\|\nu\| \leq \|\nu\|_1$, 所以, C^1 模下的收敛可以保证 C^0 模下的收敛, 故在 C^0 模下也是完备的). 因此, 只需要证明存在 $\delta_* \leq \delta_3$, 使得在 $R(\delta_*)$ 上 T 为压缩映射, 从而, 不动点问题有解.

令 $\nu^{(1)}, \nu^{(2)} \in \Sigma(\delta_3)$, 则 $u^{(1)} = T\nu^{(1)}$, $u^{(2)} = T\nu^{(2)} \in \Sigma(\delta_3)$. 设 $\nu^* = \nu^{(1)} - \nu^{(2)}$, $u^* = u^{(1)} - u^{(2)}$, 并且

$$\begin{cases} \tilde{\lambda}_l^{(\alpha)}(t, \theta) = \lambda_l^{(\alpha)}(t, \theta, \nu^{(\alpha)}(t, \theta)), \\ \tilde{\lambda}_l^{(\alpha)}(t, \theta) = \lambda_l^{(\alpha)}(t, \theta, \nu^{(\alpha)}(t, \theta)), \quad \alpha = 1, 2, \end{cases}\quad (4.51)$$

则有

$$\begin{cases} \frac{\partial u_k^*}{\partial t} + \tilde{\lambda}_k^{(1)}(t, \theta) \frac{\partial u_k^*}{\partial \theta} = \tilde{h}_k(t, \theta), \\ t = 0 : u_k^* = 0, \quad k = 1, \dots, 5, \end{cases} \quad (4.52)$$

其中

$$\tilde{h}_k = -(\tilde{\lambda}_k^{(1)} - \tilde{\lambda}_k^{(2)}) \frac{\partial u_k^{(2)}}{\partial \theta} + (\tilde{\mu}_k^{(1)} - \tilde{\mu}_k^{(2)}), \quad k = 1, \dots, 5.$$

在 $R(\delta)$ 上, 计算可得

$$\|\tilde{h}\| \leq (\Omega_1 + 3\Omega^2 + 2\max\{|c|, 1\}\Omega + |c| + 1)\|\nu^*\|. \quad (4.53)$$

由第一先验估计, 可得

$$\|u^*\| \leq (\Omega_1 + 3\Omega^2 + 2\max\{|c|, 1\}\Omega + |c| + 1)\delta\|\nu^*\|. \quad (4.54)$$

于是, 取

$$\begin{aligned} \delta_4 &= \frac{1}{3(\Omega_1 + 3\Omega^2 + 2\max\{|c|, 1\}\Omega + |c| + 1)}, \\ \Omega_1 &= 2\|\phi\| + 2\Omega^3 + 2\max\{|c|, 1\}\Omega^2 + 2|c|\Omega + 2(\|\dot{\phi}\| + 1)(\Omega + 1) \\ &= (4\|\phi\| + 2)\|\dot{\phi}\| + (16\|\phi\|^3 + 8\max\{|c|, 1\}\|\phi\|^2 + 4|c|\|\phi\| + 6\|\phi\| + 2). \end{aligned} \quad (4.55)$$

令 $D_0(\|\phi\|) = 48\|\phi\|^3 + (36 + 24\max\{|c|, 1\})\|\phi\|^2 + (12|c| + 12\max\{|c|, 1\} + 18)\|\phi\| + 30\|\phi\| + 3|c| + 9$, $D_1(\|\phi\|) = 6(2\|\phi\| + 1)$, 则有

$$\delta_4 = \frac{1}{D_0 + D_1\|\dot{\phi}\|},$$

那么, 当选取 $\delta_* = \min\{\delta_3, \delta_4\}$ 时, 算子 T 是 $R(\delta_*)$ 上的压缩映射.

综上所述可知, $\delta_* = \min\{\delta_3, \delta_4\}$, 又因为 $C_0(\|\phi\|) > D_0(\|\phi\|)$, $C_1(\|\phi\|) > D_1(\|\phi\|)$, 于是, $\delta_2 < \delta_4$. 因此, Cauchy 问题的局部解的存在区间的下界 δ_* 为

$$\delta_* = \min \left\{ \frac{\|\phi\|}{A_0 - \|\phi\|}, \frac{A_0 + A_1\|\dot{\phi}\|}{B_0 + B_1\|\dot{\phi}\| + B_2\|\dot{\phi}\|^2}, \frac{1}{C_0 + C_1\|\dot{\phi}\|} \right\}. \quad (4.56)$$

定理 1.2 得证. $\square$

注 4.1 当 $c = 0$ 时, 我们可得到不加外力项的双曲平均曲率流 Cauchy 问题的解的存在区间的下界.

注 4.2 通过 δ_* 的表达式, 即使给予的初始条件的 C^1 范数足够小, 并不能得到 $\delta_* \rightarrow \infty$. 这说明小初值并不能够保证 C^1 解在较大的区域内存在. 原因在于 Riemann 不变量形式的拟线性双曲方程组的最后一个方程中有一个常数项 1 存在, 从而在积分过程中, 小初值的性质无法保持.

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The lifespan of classical solution to the Cauchy problem for the hyperbolic mean curvature flow with a linear forcing term

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Abstract We investigate the hyperbolic mean curvature flow with a linear forcing term. By means of support function, a hyperbolic Monge-Ampère equation is derived and this equation could be reduced to a system in Riemann invariants. Using the theory of the local solution to Cauchy problem for quasilinear hyperbolic system, we discuss the lifespan (the maximal existence time of unique local classical solutions) of classical solution to the Cauchy problem for the hyperbolic mean curvature flow with a linear forcing term.

Keywords a linear forcing term, hyperbolic mean curvature flow, hyperbolic Monge-Ampère equation, Riemann invariants, lifespan

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