三维旋转 Navier-Stokes 方程有限元维数分裂方法

献给林群教授 80 华诞

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摘要 叶轮机内部三维黏性流动服从三维旋转 Navier-Stokes 方程, 流动通道被两张叶片和轮盘轮盖所包围, 有进出口, 在它上面流动满足自然边界条件. 为了克服复杂边界形状和高 Reynolds 数带来的困难. 本文采用 R -坐标系, 在旋转方向用一系列二维流形, 将流道分割为 N 个相同的流道, 在旋转方向上, 速度用一维的三次 Hermite 型有限元, 压力用一维一次 Lagrange 型的有限元逼近, 将旋转 Navier-Stokes 方程限制在 N 个流形上, 得到一个包含 $8(N+1)$ 个未知二维函数的非线性的偏微分方程系统, 它有 $N+1$ 组, 每组有 8 个未知二维函数所应满足的拟旋转 Navier-Stokes 方程. 尤其在正、负叶片面上, 得到关于流体速度法向导数和压力的偏微分方程, 称为“边界层方程”, 其他 $N-1$ 个二维流形上, 包含流体速度和压力所满足的偏微分方程. 由于速度具有三个分量和两个自变量, 故称为 2C-3D N-S 边值问题. 而对应的人工交界面 (二维流形), 称为具有可穿透和可滑动的“柔性边界”. 同时构造自然的、结构性的二度并行算法, 大大节约信息交换的次数, 节约计算时间. 作为边界层方程的求解变量包含叶片面上的法向应力, 可用于叶轮功率和叶片几何形状优化控制.

关键词 旋转 Navier-Stokes 方程 有限元维数分裂方法 边界层方程

MSC (2010) 主题分类 65M30, 76D05, 78A10

1 引言

众所周知, 叶轮内部三维黏性流动, 由于边界几何形状复杂性、混合型边界条件和高 Reynolds 数, 流动极为复杂. 现今通用软件所用的算法, 如有限体积法等, 计算精度缺失, 无法计算边界层内的速度梯度和法向应力, 从而也无法应用传统的共轭梯度法求解最优几何形状的叶片. 这个难题一直困扰科学家和工程师们. 本文提出的方法, 不但能克服上述缺点和困难, 而且给出一个具有自然结构的二度并行算法, 不但方法本身包含并行性, 而且算法与流动特性相协调. 由于方法包含边界层方程, 不但给出

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边界层的法向应力, 而且还给出阻力泛函计算的简便公式, 为叶片几何形状的最佳设计提供一个新的方法.

这个方法是建立在一个 R - 坐标系的基础之上. 这个坐标系的第三个坐标是沿着旋转方向的贯截变量. 在 R - 坐标系里, Navier-Stokes 方程分裂为一个膜算子和一个弯曲算子之和. 弯曲算子包含贯截变量和关于它的导数. 对于贯截变量, 可用差分或 Taylor 展开, 得到各不相同的维数分裂方法. 关于这些方法的构造、讨论和分析可参见文献 [1-8]. 本文建立在有限元方法之上.

2 叶轮叶片几何和一维有限元分割

图 1 是叶轮叶片的图形, 图 2 是叶轮和流道的图形.

图 1 叶片

图 2 叶轮和流道

假设叶轮叶片 $\mathfrak{S}$ 的厚度充分小使得可以忽略不计. 叶轮内的通道记为 Ω_ε , 它由两个相连的叶片 Γ_s^+, Γ_s^- 和轮盘轮盖 Γ_t, Γ_b 所包围, 进出口有人工边界、入流边界 Γ_{in} 和出流边界 Γ_{out} (见图 1 和 2),

$$\partial\Omega_\varepsilon = \Gamma_0 \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \quad \Gamma_0 = \Gamma_t \cup \Gamma_b \cup \Gamma_s^+ \cup \Gamma_s^-,$$

Γ_0 是固壁边界. 本文采用以叶轮旋转轴为 z 轴的旋转圆柱坐标系 (r, θ, z) , 它是非惯性坐标系. 叶片中心面是一个二维流形 $\mathfrak{S}$, 为简便, 描述它的 Gauss 坐标系用 $(x^1 = z, x^2 = r)$, 由于曲面是一个光滑的浸入 $\mathfrak{R} \in C^3(D, \mathbf{E}^3)$, 即存在一个光滑函数 $\Theta(x^1, x^2)$ 使得

$$D = \{(x^1, x^2) \in \mathbf{E}^2\} \rightarrow \mathbf{E}^3, \quad \mathfrak{R}(x^1, x^2) = x^2 \mathbf{e}_r + r\Theta(x^1, x^2) \mathbf{e}_\theta + x^1 \mathbf{k},$$

其中 $D \subset \mathbf{E}^2$ 是 $\mathbf{E}^2$ 中的单连通域, 是叶片在子午面上的投影, 它的边界为

$$\partial D = \gamma_0 \cup \gamma_{\text{in}} \cup \gamma_{\text{out}}, \quad \gamma_0 = \gamma_t \cup \gamma_b,$$

这里 $\gamma_b, \gamma_t, \gamma_{\text{in}}, \gamma_{\text{out}}$ 分别是 $\Gamma_b, \Gamma_t, \Gamma_{\text{in}}, \Gamma_{\text{out}}$ 在子午面上的投影. $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k})$ 是旋转圆柱坐标系的基向量. $\mathfrak{R}(x^\alpha)$ 是 $\mathfrak{S}$ 上在 x^α 点的向径, 而 $\mathfrak{S}$ 上 Gauss 坐标系的基向量

$$\mathbf{e}_\alpha = \frac{\partial \mathfrak{R}}{\partial x^\alpha} = \delta_{2\alpha} \mathbf{e}_r + r\Theta_\alpha \mathbf{e}_\theta + \delta_{1\alpha} \mathbf{k}, \quad (2.1)$$

其中 $\Theta_\alpha = \frac{\partial \Theta}{\partial x^\alpha}$. 曲面 $\mathfrak{S}$ 在 Gauss 坐标系 (x^α) 下的度量张量为

$$\begin{cases} a_{\alpha\beta} = \mathfrak{R}_\alpha \cdot \mathfrak{R}_\beta = \delta_{\alpha\beta} + r^2 \Theta_\alpha \Theta_\beta, \\ a = \det(a_{\alpha\beta}) = 1 + r^2 |\nabla \Theta|^2 > 0, \quad |\nabla \Theta|^2 = \Theta_1^2 + \Theta_2^2, \end{cases} \quad (2.2)$$

$\mathfrak{S}$ 上的法向量

$$\mathbf{n} = \frac{\mathfrak{R}_1 \times \mathfrak{R}_2}{|\mathfrak{R}_1 \times \mathfrak{R}_2|} = \frac{-\Theta_2 \mathbf{e}_r + \mathbf{e}_\theta - \Theta_1 \mathbf{k}}{\sqrt{a}}. \quad (2.3)$$

本文还要引入循环张量, $\varepsilon_{\alpha\beta}$ 和 $\varepsilon^{\alpha\beta}$ 分别记循环张量的协变分量和逆变分量,

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, \\ -\sqrt{a}, \\ 0, \end{cases} \quad \varepsilon^{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{a}}, & (\alpha, \beta) : (1, 2) \text{ 的偶排列}, \\ -\frac{1}{\sqrt{a}}, & (\alpha, \beta) : (1, 2) \text{ 的奇排列}, \\ 0, & \text{其他情形}. \end{cases}$$

另外, $\mathfrak{S}$ 上的第二基本型系数, 即曲面的曲率张量, 以及平均曲率张量和 Gauss 曲率张量分别为

$$\begin{cases} b_{\alpha\beta} = \mathbf{n} \mathfrak{R}_{\alpha\beta} = \frac{1}{\sqrt{a}} [\delta_{2\alpha} \Theta_\beta + \delta_{2\beta} \Theta_\alpha + r^2 \Theta_2 \Theta_\alpha \Theta_\beta + r \Theta_{\alpha\beta}], \\ b = \det(b_{\alpha\beta}), \quad K = \frac{b}{a}, \quad 2H = a^{\alpha\beta} b_{\alpha\beta} = \varepsilon^{\alpha\beta} \varepsilon^{\lambda\sigma} b_{\alpha\lambda} a_{\beta\sigma}. \end{cases} \quad (2.4)$$

设 N 是叶轮的叶片数, $\varepsilon = \pi/N$. 两个相邻叶片之间的展角为 $\frac{2\pi}{N}$. 负压面对应于

$$\theta_- = -\frac{\pi}{N} + \Theta(x^\alpha),$$

而正压面对应于

$$\theta_+ = \frac{\pi}{N} + \Theta(x^\alpha).$$

显然, $\theta = 0$ 对应于流道 Ω_ε 的中心面. 流道展角为 $\theta_+ - \theta_- = \frac{2\pi}{N} = 2\varepsilon$. 容易证明, 存在一个单参数曲面族 $\mathfrak{S}_\xi$ 覆盖流道 $\Omega_\varepsilon: D \rightarrow \mathfrak{S}_\xi = \{\mathbf{R}(x^1, x^2; \xi) : \forall (x^1, x^2) \in D\}$, 使得

$$\mathfrak{R}(x^1, x^2; \xi) = x^2 \mathbf{e}_r + x^2(\varepsilon \xi + \Theta(x^1, x^2)) \mathbf{e}_\theta + x^1 \mathbf{k}.$$

显然, 对任何固定的 ξ , 曲面 $\mathfrak{S}_\xi$ 的度量张量 $a_{\alpha\beta}$ 与 (2.2) 一致, 它是非奇异的, 且与 ξ 无关. 由此可以定义一个三维 Euclid 空间 E^3 中的曲线坐标系 (x^1, x^2, ξ) ,

$$(r, \theta, z) \rightarrow (x^1, x^2, \xi) : x^1 = z, \quad x^2 = r, \quad \xi = \varepsilon^{-1}(\theta - \Theta(x^1, x^2)), \quad (2.5)$$

称为 R -坐标系. 流道

$$\Omega_\varepsilon = \{\mathbf{R}(x^1, x^2, \xi) = x^2 \mathbf{e}_r + x^2(\varepsilon \xi + \Theta(x^1, x^2)) \mathbf{e}_\theta + x^1 \mathbf{k}, \forall (x^1, x^2, \xi) \in \Omega\}$$

变换为 E^3 (三维 Euclid 空间) 中的一个固定区域 $\Omega = \{(x^1, x^2) \in D, -1 \leq \xi \leq 1\} \subset \mathbb{R}^3$, 它与叶片的曲面 $\mathfrak{S}$ 无关, 变换的 Jacobi 行列式

$$J\left(\frac{\partial(r, \theta, z)}{\partial(x^1, x^2, \xi)}\right) = \varepsilon,$$

它是非奇异的. 记 $(x^{1'}, x^{2'}, x^{3'}) = (r, \theta, z)$, E^3 在圆柱坐标系下的度量张量记为 $(g_{1'1'} = 1, g_{2'2'} = r^2, g_{3'3'} = 1, g_{i'j'} = 0, \forall i' \neq j')$. 根据张量变换规律,

$$g_{ij} = g_{i'j'} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j}.$$

在新坐标系下, E^3 的度量张量的协变分量和逆变分量分别为 (参见文献 [9–11])

$$\begin{cases} g_{\alpha\beta} = a_{\alpha\beta}, & g_{3\beta} = g_{\beta 3} = \varepsilon r^2 \Theta_\beta, & g_{33} = \varepsilon^2 r^2, & g = \det(g_{ij}) = \varepsilon^2 r^2, \\ g^{\alpha\beta} = \delta^{\alpha\beta}, & g^{3\beta} = g^{\beta 3} = -\varepsilon^{-1} \Theta_\beta, & g^{33} = \varepsilon^{-2} r^{-2} (1 + r^2 |\nabla \Theta|^2) = (r\varepsilon)^{-2} a. \end{cases} \quad (2.6)$$

值得特别指出的是, 这个度量张量与 ξ 无关.

贯截变量的区间 $[-1, 1]$ 被分割为 m 个元素,

$$[-1, 1] = \sum_{k=0}^{m-1} [\xi_k, \xi_{k+1}], \quad \Delta \xi = \xi_{k+1} - \xi_k = \delta,$$

共有 $m+1$ 个节点: $\xi_0 = -1, \xi_1, \dots, \xi_m = 1$, 每个节点 $\xi_k = \text{constant}$ 对应于一个二维流形 $\mathfrak{S}_k \subset \mathbb{R}^3$, 它是三维 Euclid 空间一个子流形 (见图 3 和 4), 它们同构于叶片面 $\mathfrak{S}$ (微分几何基本定理^[1, 9, 10]). 对速度空间, 本文采用有 4 个自由度的三次 Hermite 型一维元素^[11], 每个节点有两个自由度: 函数值和它的一阶导数值. Hermite 形状函数是

图 3 流道 Ω_ε 在子午平面上的截图 D

图 4 流道的剖分图

$$\begin{cases} L_1^{(0)}(\eta) = \frac{1}{4}(\eta-1)^2(2+\eta), \\ L_2^{(0)}(\eta) = \frac{1}{4}(\eta+1)^2(2-\eta), \\ L_1^{(1)}(\eta) = \frac{1}{4}(\eta-1)^2(\eta+1), \\ L_2^{(1)}(\eta) = \frac{1}{4}(\eta+1)^2(\eta-1), \end{cases} \quad (2.7)$$

η 是 $[-1, 1]$ 上的自然坐标, 它和物理坐标的仿射变换为

$$\xi = \frac{\delta}{2}(\eta+1) + \xi_k, \quad \delta = \xi_{k+1} - \xi_k, \quad -1 \leq \eta \leq 1 \Rightarrow \xi_k \leq \xi \leq \xi_{k+1}, \quad \int_{\xi_k}^{\xi_{k+1}} d\xi = \frac{\delta}{2} \int_{\eta=-1}^{\eta=1} d\eta, \quad (2.8)$$

压力所在的空间, 用一维 Lagrange 型的形状函数

$$L_1(\eta) = \frac{1}{2}(1-\eta), \quad L_2(\eta) = \frac{1}{2}(1+\eta).$$

为简便, Hermite 形状函数 (2.7) 记为

$$\begin{cases} H_1(\eta) = L_1^{(0)}(\eta) = \frac{1}{4}(\eta-1)^2(2+\eta), \\ H_3(\eta) = L_2^{(0)}(\eta) = \frac{1}{4}(\eta+1)^2(2-\eta), \\ H_2(\eta) = L_1^{(1)}(\eta) = \frac{1}{4}(\eta-1)^2(\eta+1), \\ H_4(\eta) = L_2^{(1)}(\eta) = \frac{1}{4}(\eta+1)^2(\eta-1), \end{cases} \quad (2.9)$$

而形状函数的张量积

$$\begin{cases} \psi_{ij}(\xi) = H_i(\xi)H_j(\xi), & i, j = 1, 2, 3, 4, \\ \psi_{ij}^{(1)}(\xi) = H_i(\xi)H_j'(\xi), \quad \psi_{ij}^{(2)}(\xi) = H_i'(\xi)H_j(\xi), \quad \psi_{ij}^{(3)}(\xi) = H_i'(\xi)H_j'(\xi), & i, j = 1, 2, 3, 4, \\ \varphi_{\alpha\beta}(\xi) = L_\alpha(\xi)L_\beta(\xi), & \alpha, \beta = 1, 2, \end{cases} \quad (2.10)$$

其中

$$L_1'^{(0)}(\xi) := \frac{\partial L_1^{(0)}}{\partial \xi}(\xi), \quad \dots$$

本文采用记号

$$\begin{cases} \alpha_{ij} = \int_{\xi_k}^{\xi_{k+1}} \psi_{ij} d\eta = \frac{\delta}{2} \int_{-1}^1 \psi_{ij}(\xi) d\xi, & \alpha_{ij}^{(i)} = \frac{\delta}{2} \int_{-1}^1 \psi_{ij}^{(i)}(\xi) d\xi, \quad i = 1, 2, 3, \\ \{\pi_{\alpha\beta}\} = \left\{ \int_{\xi_k}^{\xi_{k+1}} \varphi_{\alpha\beta} d\eta \right\} = \begin{bmatrix} \frac{\delta}{3} & \frac{\delta}{6} \\ \frac{\delta}{6} & \frac{\delta}{3} \end{bmatrix}, \\ \alpha_{ij,l} = \int_{\xi_k}^{\xi_{k+1}} \psi_{ij}(\xi) H_l(\xi) d\xi = \frac{\delta}{2} \int_{-1}^1 \psi_{ij} H_l(\eta) d\eta, & \alpha_{ij,l}^{(1)} = \frac{\delta}{2} \int_{-1}^1 \psi_{ij}^{(1)} H_l(\eta) d\eta, \\ \gamma_{i\alpha} = \frac{\delta}{2} \int_{-1}^1 H_i(\eta) L_\alpha(\eta) d\eta, & \gamma_{i\alpha}^{(1)} = \frac{\delta}{2} \int_{-1}^1 H_i'(\eta) L_\alpha(\eta) d\eta, \end{cases} \quad (2.11)$$

以及

$$(\cdot, \cdot) = \int_D \cdot \times \cdot r \varepsilon dx.$$

在 $[\xi_k, \xi_{k+1}]$ 上的速度场 $\mathbf{w}$ 和压力场 p 的有限元插值函数分别为

$$\begin{aligned} \mathbf{w}_h(x, \xi) &\doteq \sum_{i=1}^4 H_i(\xi) \mathbf{w}_k(i), & \mathbf{v}_h(x, \xi) &= \sum_{i=1}^4 H_i(\xi) \mathbf{v}_k(i), & \text{在元素 } [\xi_k, \xi_{k+1}] \text{ 上,} \\ p_h(x, \xi) &= \sum_{\alpha=1}^2 L_\alpha(\xi) p_k(\alpha), & & \text{在元素 } [\xi_k, \xi_{k+1}] \text{ 上,} \end{aligned} \quad (2.12)$$

其中

$$\begin{cases} \mathbf{w}_k(1) = \mathbf{w}(x, \xi_k), & \mathbf{w}_k(3) = \mathbf{w}(x, \xi_{k+1}), \\ \mathbf{w}_k(2) = \frac{\partial \mathbf{w}}{\partial \xi} \Big|_{\mathfrak{S}_k}, & \mathbf{w}_k(4) = \frac{\partial \mathbf{w}}{\partial \xi} \Big|_{\mathfrak{S}_{k+1}}, \\ \mathbf{v}_k(2) = \frac{\partial \mathbf{v}}{\partial \xi} \Big|_{\mathfrak{S}_k}, & \mathbf{v}_k(4) = \frac{\partial \mathbf{v}}{\partial \xi} \Big|_{\mathfrak{S}_{k+1}}, \\ p_k(1) = p(x, \xi_k), & p_k(2) = p(x, \xi_{k+1}). \end{cases} \quad (2.13)$$

以后在不引起混淆的情形, $w_k := w_k(x)$, $u_k = u_k(x)$, ... 等, 对应于 $\mathfrak{S}_k$ 和 $\mathfrak{S}_{k+1}$ 上的 2D-3C 速度场 (如二维三个分量) 和压力场. 同时为简便, 记

$$\mathbf{W}_k = \{\mathbf{w}_k(i), i = 1, 2, 3, 4\}, \quad \mathbf{V}_k = \{\mathbf{v}_k(i), i = 1, 2, 3, 4\}, \quad P_k = \{p_k(\alpha), \alpha = 1, 2\}. \quad (2.14)$$

前面已讲过, 流道 Ω_ε 在 R -坐标变换下变成 $\Omega = D \times \{-1, 1\}$, 后者可以分割成为

$$\Omega = \sum_k \Omega_k, \quad \Omega_k = D \times \{\xi_i, \xi_{i+1}\}, \quad -1 = \xi_0 < \xi_1 < \cdots < \xi_m = 1,$$

m 个子区域 $\Omega_k = D \times \{\xi_k, \xi_{k+1}\}$ (称为流层) 之并 (见图 4). 任一个节点 $\xi = \varepsilon^{-1}(\theta - \Theta(x^1, x^2)) =$ 常数, 对应一个二维流形, 记为 $\mathfrak{S}_\xi$. 由 $\mathfrak{S}_{\xi_k}, \mathfrak{S}_{\xi_{k+1}}, \Gamma_t, \Gamma_b$ 围成的子区域 $\hat{\Omega}_k$ 被影射为 Ω_k . 为简便, $\mathfrak{S}_{\xi_k}$ 记为 $\mathfrak{S}_k$.

最后, Ω_ε 等于 $\hat{\Omega}_k$ 之并 $\Omega_\varepsilon = \bigcup_{k=0}^m \hat{\Omega}_k$, 它被 $m+1$ 个二维流形 $\mathfrak{S}_k$, $k = 0, 1, 2, \dots$ 所分割, $\mathfrak{S}_0, \mathfrak{S}_{m+1}$ 对应正负叶片面, 在它上面, 速度满足非滑动边界条件 $\mathbf{w} = \mathbf{0}$; 而在 $\mathfrak{S}_k$, $k = 1, 2, \dots, m$ 上面, 流体可以穿透它, 如 $\mathbf{w}|_{\mathfrak{S}_k} \neq \mathbf{0}$, $k = 1, 2, \dots, m$. 交界面 $\mathfrak{S}_k$ 称为柔性曲面. $\mathfrak{S}_k$ 和 $\mathfrak{S}_{k+1}$ 之间夹角 (见图 4) 为

$$\theta_{k+1} - \theta_k = \varepsilon(\xi_{k+1} - \xi_k) = \frac{\pi}{N} \Delta \xi_k, \quad \delta := \Delta \xi_k = \xi_{k+1} - \xi_k = \frac{2}{m}.$$

本文的目的就是求流层 $\Omega_{k,k+1}$ 内的旋转 Navier-Stokes 的逼近解 $(\mathbf{w}_k, p_k)$.

3 旋转 Navier-Stokes 的一维有限元逼近

我们知道, 叶轮机内部流动服从旋转 Navier-Stokes 边值问题

$$\begin{cases} \nabla_j e^{ij}(\mathbf{w}) + w^j \nabla_j w^i + 2(\vec{\omega} \times \mathbf{w})^i - \nabla^i p = f^i, & \text{在 } \Omega_\varepsilon \text{ 内,} \\ \operatorname{div} \mathbf{w} = 0, & \text{在 } \Omega_\varepsilon \text{ 内,} \\ \mathbf{w}|_{\Gamma_0} = 0, \quad (\mathbf{w})|_{\Gamma_{\text{in}}} = (\mathbf{w}_{\text{in}}), \quad \sigma \cdot \mathbf{n}|_{\Gamma_{\text{out}}} = \mathbf{h}, \end{cases} \quad (3.1)$$

其中 $\vec{\omega}$ 为叶轮旋转角速度向量, 流体速度的协变导数和速度变形速率张量为

$$\begin{cases} e^{ij}(\mathbf{w}) = g^{ik} g^{jm} e_{km}(\mathbf{w}), \quad e_{ij}(\mathbf{w}) = \frac{1}{2}(g_{il} \nabla_j w^l + g_{jl} \nabla_i w^l), \\ \nabla_i w^j = \partial_i w^j + \Gamma_{ik}^j w^k, \quad \operatorname{div} \mathbf{w} = \nabla_i w^i, \end{cases} \quad (3.2)$$

Γ_{jk}^i 为 R -坐标系 (x^α, ξ) 下的 Christoffel 记号. 外力为重力和离心力, 本文不计重力, 那么, Coriolis 力 $\mathbf{C}$ 和离心力 $\mathbf{f}$ 分别为 (参见文献 [12–14])

$$\begin{cases} \mathbf{C} := 2\vec{\omega} \times \mathbf{w} = C^i e_i, \\ C^\alpha(\mathbf{w}) = -2\omega r \delta_2^\alpha (\Theta_\lambda w^\lambda + \varepsilon w^3), \quad C^3(\mathbf{w}) = 2\omega(r\varepsilon)^{-1}(a_{2\lambda} w^\lambda + r^2 \Theta_2 w^3), \\ C_\alpha = g_{\alpha k} C^k = 2\omega r ((\Theta_\alpha \delta_{2\lambda} - \Theta_\lambda \delta_{2\alpha}) w^\lambda - \delta_{2\alpha} \varepsilon w^3), \quad C_3 = g_{3k} C^k = 2\omega r \varepsilon \delta_{2\lambda} w^\lambda, \\ \mathbf{f} = \omega^2 r \mathbf{e}_r = (\omega)^2 r (\mathbf{e}_2 - \varepsilon^{-1} \Theta_2 \mathbf{e}_3), \quad f^1 = 0, \quad f^2 = (\omega)^2 r, \quad f^3 = -\varepsilon^{-1} (\omega)^2 r \Theta_2. \end{cases} \quad (3.3)$$

命题 1 ^[1,2] 在 R -坐标系 (x^α, ξ) 下, Christoffel 记号和一阶张量的协变导数分别为

$$\begin{cases} \Gamma_{\beta\gamma}^\alpha = -r \delta_{2\alpha} \Theta_\beta \Theta_\gamma, \quad \Gamma_{3\beta}^\alpha = -\varepsilon r \delta_{2\alpha} \Theta_\beta, \quad \Gamma_{3\alpha}^3 = \Gamma_{\alpha 3}^3 = r^{-1} a_{2\alpha}, \quad \Gamma_{33}^\alpha = -\varepsilon^2 r \delta_{2\alpha}, \\ \Gamma_{\alpha\beta}^3 = \varepsilon^{-1} r^{-1} (\delta_{2\alpha} \Theta_\beta + \delta_{2\beta} \Theta_\alpha + r^2 \Theta_2 \Theta_\alpha \Theta_\beta) + \varepsilon^{-1} \Theta_{\alpha\beta}, \quad \Gamma_{33}^3 = \varepsilon r \Theta_2, \\ \nabla_\alpha w^\beta = \frac{\partial w^\beta}{\partial x^\alpha} - r \delta_{2\beta} \Theta_\alpha \Pi(\mathbf{w}, \Theta), \\ \nabla_\alpha w^3 = \partial_\alpha w^3 + \varepsilon^{-1} \Theta_{\alpha\lambda} w^\lambda + (r\varepsilon)^{-1} [\Theta_\alpha w^2 + a_{2\alpha} \Pi(\mathbf{w}, \Theta)], \\ \nabla_3 w^\beta = \frac{\partial w^\beta}{\partial \xi} - r \varepsilon \delta_{2\beta} \Pi(\mathbf{w}, \Theta), \quad \nabla_3 w^3 = \frac{\partial w^3}{\partial \xi} + \frac{w^2}{r} + r \Theta_2 \Pi(\mathbf{w}, \Theta), \\ \operatorname{div} \mathbf{w} = \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{w^2}{x^2} + \frac{\partial w^3}{\partial \xi}, \quad \Pi(\mathbf{w}, \Theta) = \varepsilon w^3 + w^\beta \Theta_\beta. \end{cases} \quad (3.4)$$

命题 2 ^[1,2] 在 R -坐标系 (x^α, ξ) 下, 变形速度张量 $e_{ij}(\mathbf{w})$ 分裂为膜算子和弯曲算子之和,

$$\begin{cases} e_{ij}(\mathbf{w}) = \frac{1}{2} [g_{ik} \nabla_j w^k + g_{jk} \nabla_i w^k], \\ e_{\alpha\beta}(\mathbf{w}) = E_{\alpha\beta\sigma}^\lambda \frac{\partial w^\sigma}{\partial x^\lambda} + E_{\alpha\beta 3}^\lambda \frac{\partial w^3}{\partial x^\lambda} + E_{\alpha\beta\sigma} w^\sigma, \\ e_{3\alpha}(\mathbf{w}) = e_{3\alpha}^*(\mathbf{w}) + \frac{1}{2} a_{\alpha\lambda} \frac{\partial w^\lambda}{\partial \xi} + \frac{1}{2} \varepsilon r^2 \Theta_\alpha \frac{\partial w^3}{\partial \xi}, \\ e_{3\alpha}^*(\mathbf{w}) = E_{3\alpha\sigma}^\lambda \frac{\partial w^\sigma}{\partial x^\lambda} + E_{3\alpha 3}^\lambda \frac{\partial w^3}{\partial x^\lambda} + E_{3\alpha\sigma} w^\sigma, \\ e_{33}(\mathbf{w}) = e_{33}^*(\mathbf{w}) + \varepsilon^2 r^2 \frac{\partial w^3}{\partial \xi} + \varepsilon r^2 \frac{\partial w^\lambda}{\partial \xi} \Theta_\lambda, \\ e_{33}^*(\mathbf{w}) = E_{33\sigma} w^\sigma = \varepsilon^2 r \delta_{2\sigma} w^\sigma, \end{cases} \quad (3.5)$$

其中

$$\begin{cases} E_{\alpha\beta\sigma}^\lambda = \frac{1}{2}(a_{\alpha\sigma}\delta_{\beta\lambda} + a_{\beta\sigma}\delta_{\alpha\lambda}), & E_{\alpha\beta 3}^\lambda = \frac{1}{2}\varepsilon r^2(\delta_{\alpha\sigma}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\sigma})\Theta_\sigma, \\ E_{\alpha\beta\sigma} = \frac{1}{2}r^2(\Theta_\beta\Theta_{\alpha\sigma} + \Theta_\alpha\Theta_{\beta\sigma}) + r\Theta_\alpha\Theta_\beta\delta_{2\sigma}, & E_{3\alpha\sigma}^\lambda := \frac{1}{2}r^2\varepsilon\Theta_\sigma\delta_{\alpha\lambda}, \\ E_{3\alpha 3}^\lambda := \frac{1}{2}r^2\varepsilon^2\delta_{\alpha\lambda}, & E_{3\alpha\sigma} = \frac{1}{2}r\varepsilon(2\Theta_\alpha\delta_{2\sigma} + r\Theta_{\sigma\alpha}), & E_{33\sigma} = \varepsilon^2r\delta_{2\sigma}. \end{cases} \quad (3.6)$$

非线性项也可分解为膜算子和弯曲算子之和,

$$\begin{cases} B^\alpha(\mathbf{w}, \mathbf{w}) = w^j \nabla_j w^\alpha = w^\beta \frac{\partial w^\alpha}{\partial x^\beta} + \pi_{ij}^\alpha w^i w^j + w^3 \frac{\partial w^\alpha}{\partial \xi}, \\ B^3(\mathbf{w}, \mathbf{w}) = w^j \nabla_j w^3 = w^\beta \frac{\partial w^3}{\partial x^\beta} + \pi_{ij}^3 w^i w^j + w^3 \frac{\partial w^3}{\partial \xi}, \end{cases} \quad (3.7)$$

其中

$$\begin{aligned} \pi_{\lambda\sigma}^\alpha &= \Theta_\lambda\Theta_\sigma, & \pi_{3\lambda}^\alpha &= r\varepsilon\delta_{2\alpha}\Theta_\lambda, & \pi_{33}^\alpha &= \delta_{2\alpha}r\varepsilon^2, & \pi_{33}^3 &= r\varepsilon\Theta_2, \\ \pi_{\lambda\sigma}^3 &= \varepsilon^{-1}[\Theta_{\lambda\sigma} + r\Theta_2\Theta_\lambda\Theta_\sigma + 2r^{-1}\delta_{2\sigma}\Theta_\lambda], & \pi_{3\lambda}^3 &= r\varepsilon\Theta_2 + \frac{1}{2}\delta_{2\lambda}r^{-1}. \end{aligned} \quad (3.8)$$

命题 3 ^[1,2] 在 R - 坐标系 $\{x^\alpha, \xi\}$ 下的旋转 Navier-Stokes 方程可以表示为

$$\begin{cases} \mathcal{L}^k(\mathbf{w}, \Theta) + C^k(\omega, \mathbf{w}) + g^{kj}\partial_j p + N^k(\mathbf{w}, \mathbf{w}) = f^k, \\ \frac{\partial w^\alpha}{\partial x^\alpha} + \frac{\partial w^3}{\partial \xi} + \frac{w^2}{r} = 0, \end{cases} \quad (3.9)$$

其中

$$\begin{cases} \mathcal{L}^k(\mathbf{w}, \Theta) := -\nu \left[\tilde{\Delta} w^k + (r\varepsilon)^{-2} a \frac{\partial^2 w^k}{\partial \xi^2} - 2\varepsilon^{-1} \Theta_\beta \frac{\partial^2 w^k}{\partial x^\beta \partial \xi} + L_m^{kj} \frac{\partial w^m}{\partial x^j} + q_m^k w^m \right], \\ C^k(\omega, \mathbf{w}) = C_\lambda^k w^\lambda + C_3^k w^3, \\ N^k(\mathbf{w}, \mathbf{w}) = w^\lambda \frac{\partial w^k}{\partial x^\lambda} + \pi_{ij}^k w^i w^j + w^3 \frac{\partial w^k}{\partial \xi}, \\ \tilde{\Delta} w^k = \frac{\partial^2 w^k}{\partial x^\alpha \partial x^\alpha}, & L_\lambda^{\alpha\beta}(\Theta) = r^{-1} \delta_{\alpha\lambda} \delta_{2\beta}, & L_3^{\alpha\beta} = 0, \\ L_\lambda^{\alpha 3}(\Theta) = -(r\varepsilon)^{-1}(\delta_{\alpha\lambda} \Theta_2 + r \delta_{\alpha\lambda} \tilde{\Delta} \Theta) - 2(r\varepsilon)^{-1} a (\Theta_2 \delta_{\alpha\lambda} + \delta_{2\alpha} \Theta_\lambda), \\ L_3^{\alpha 3} = -2r^{-1} \delta_{2\alpha}, & q_3^\alpha = -\varepsilon \delta_{2\alpha} \Theta_2, & q_\sigma^\alpha := \delta_{2\alpha} r^2 (a_{2\sigma} - r \Theta_\lambda \Theta_{\lambda\sigma}), \\ L_\lambda^{3\beta}(\Theta) = 2(r\varepsilon)^{-1} (r \Theta_{\beta\lambda} + \delta_{2\beta} \Theta_\lambda), & L_3^{3\beta}(\Theta) = (3-a)r^{-1} \delta_{2\beta}, \\ L_\lambda^{33}(\Theta) = 2\varepsilon^{-2} r^{-3} (\delta_{2\lambda} - r^3 \Theta_\beta \Theta_{\beta\lambda}), & L_3^{33}(\Theta) = -(r\varepsilon)^{-1} (\Theta_2 + r \tilde{\Delta} \Theta), \\ q_\sigma^3 := \varepsilon^{-1} \partial_\sigma \tilde{\Delta} \Theta + 2(r\varepsilon)^{-1} a_{2\sigma} \tilde{\Delta} \Theta + 3(r\varepsilon)^{-1} \Theta_{2\sigma} + \varepsilon^{-1} r^{-2} (2\Theta_\sigma a_{22} + 3\delta_{2\sigma} \Theta_2), & q_3^3 = 0, \\ C_\beta^\alpha = -2r\omega \delta_2^\alpha \Theta_\beta, & C_3^\alpha = -2r\omega \delta_2^\alpha \varepsilon, & C_\beta^3 = 2\omega(r\varepsilon)^{-1} a_{2\beta}, & C_3^3 = 2\omega r \varepsilon^{-1} \Theta_2, \end{cases} \quad (3.11)$$

π_{jm}^i 由 (3.8) 定义.

命题 4 在 R - 坐标系 $\{x^\alpha, \xi\}$ 下, 旋转 Navier-Stokes 方程的变分问题为

$$\begin{cases} \text{求 } \mathbf{u} \in V(\Omega), p \in L^2(\Omega), \text{ 使得} \\ a(\mathbf{w}, \mathbf{v}) + (g_{ij} C^i(\mathbf{w}), v^j) + (g_{ij} B^i(\mathbf{w}, \mathbf{w}), v^j) - (p, \operatorname{div} \mathbf{v}) = \langle \mathbf{F}, \mathbf{v} \rangle, & \forall \mathbf{v} \in V(\Omega), \\ (q, \operatorname{div} \mathbf{w}) = 0, & \forall q \in L^2(\Omega), \end{cases} \quad (3.12)$$

其中

$$\left\{ \begin{aligned} a(\mathbf{w}, \mathbf{v}) &= (2\mu g^{ik} g^{jm} e_{ij}(\mathbf{w}), e_{km}(\mathbf{v})) \\ &= 2\mu[(e_{\alpha\beta}(\mathbf{w}), e_{\alpha\beta}(\mathbf{v})) + (2(r\varepsilon)^{-2}(a_{\alpha\beta} + (a-1)\delta_{\alpha\beta})e_{3\alpha}(\mathbf{w}), e_{3\beta}(\mathbf{v})) \\ &\quad + ((r\varepsilon)^{-4}a^2 e_{33}(\mathbf{w}), e_{33}(\mathbf{v}))] + 2\mu\{-2\varepsilon^{-1}((e_{\alpha\beta}(\mathbf{w}), \Theta_{\beta}e_{3\alpha}(\mathbf{v})) \\ &\quad + (e_{\alpha\beta}(\mathbf{v}), \Theta_{\beta}e_{3\alpha}(\mathbf{w}))) + (e_{\alpha\beta}(\mathbf{w}), (r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})e_{33}(\mathbf{v})) \\ &\quad + (e_{\alpha\beta}(\mathbf{v}), (r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})e_{33}(\mathbf{w})) - 2\varepsilon^{-1}(((r\varepsilon)^{-2}a\Theta_{\alpha}e_{3\alpha}(\mathbf{w}), e_{33}(\mathbf{v})) \\ &\quad + ((r\varepsilon)^{-2}a\Theta_{\alpha}e_{3\alpha}(\mathbf{v}), e_{33}(\mathbf{w})))\}, \\ B_i(\mathbf{w}, \mathbf{w}) &= g_{ij}B^j(\mathbf{w}, \mathbf{w}), \\ B_{\alpha}(\mathbf{w}, \mathbf{w}) &= a_{\alpha\beta}\left[w^{\lambda}\frac{\partial w^{\beta}}{\partial x^{\lambda}} + \pi_{ij}^{\beta}w^iw^j + w^3\frac{\partial w^{\beta}}{\partial \xi}\right] + r\varepsilon^2\Theta_{\alpha}\left[w^{\lambda}\frac{\partial w^3}{\partial x^{\lambda}} + \pi_{ij}^3w^iw^j + w^3\frac{\partial w^3}{\partial \xi}\right], \\ B_3(\mathbf{w}, \mathbf{w}) &= r^2\varepsilon\Theta_{\beta}\left[w^{\lambda}\frac{\partial w^{\beta}}{\partial x^{\lambda}} + \pi_{ij}^{\beta}w^iw^j + w^3\frac{\partial w^{\beta}}{\partial \xi}\right] + r^2\varepsilon^2\left[w^{\lambda}\frac{\partial w^3}{\partial x^{\lambda}} + \pi_{ij}^3w^iw^j + w^3\frac{\partial w^3}{\partial \xi}\right], \\ \langle \mathbf{F}, \mathbf{v} \rangle &= (\mathbf{f}, \mathbf{v}) + \langle \mathbf{h}_{\text{out}}, \mathbf{v} \rangle_{\Gamma_{\text{out}}}, \\ (\mathbf{f}, \mathbf{v}) &= (-r(\omega)^2 e_r, \mathbf{v}) = -(\omega)^2(r\delta_{\alpha 2}, v^{\alpha}) + (r^2\varepsilon^{\alpha\beta}\theta_{\alpha}\delta_{\beta 1}, v^3) = -(\omega)^2(r, v^2) - \left(\frac{r^2\Theta_2}{\sqrt{a}}, v^3\right). \end{aligned} \right. \quad (3.13)$$

引理 1 ^[1,2] 设 Θ 二阶光滑, 那么, $\forall \mathbf{w}, \mathbf{v} \in V(\Omega_{\varepsilon}) = \{\mathbf{v} \in (H^1(\Omega_{\varepsilon}))^3, \mathbf{v}|_{\Gamma_0} = 0\}$, 存在只依赖于 Θ 的常数 k_0 和 k_1 , 成立

$$a(\mathbf{w}, \mathbf{v}) = a(\mathbf{v}, \mathbf{w}), \quad |a(\mathbf{w}, \mathbf{v})| \leq k_1 \|\mathbf{w}\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega}, \quad a(\mathbf{w}, \mathbf{w}) \geq k_0 \|\mathbf{w}\|_{0,\Omega}^2,$$

其中 $\|\mathbf{w}\|_{0,\Omega}$ 与 $\|\mathbf{w}\|_{1,\Omega}$ 是等价范数, 这里

$$\begin{aligned} \|\mathbf{w}\|_{0,\Omega}^2 &= \sum_{\alpha,\beta} \|\varphi_{\alpha\beta}(\mathbf{w})\|_{0,\Omega}^2 + (r\varepsilon)^{-2} \sum_{\alpha} \|\varphi_{3\alpha}(\mathbf{w})\|_{0,\Omega}^2 + \frac{1}{2}(r\varepsilon)^{-4} \|\varphi_{33}(\mathbf{w})\|_{0,\Omega}^2, \\ \varphi_{\alpha\beta}(\mathbf{w}) &= \frac{1}{2}(\partial_{\alpha}w^{\beta} + \partial_{\beta}w^{\alpha}), \quad \varphi_{3\alpha}(\mathbf{w}) = \varphi_{\alpha 3}(\mathbf{w}) = \frac{1}{2}\left(\frac{\partial w^{\alpha}}{\partial \xi} + r^2\varepsilon^2\frac{\partial w^3}{\partial x^{\alpha}}\right), \\ \varphi_{33}(\mathbf{w}) &= r^2\varepsilon^2\left(\frac{\partial w^3}{\partial \xi} + \frac{w^2}{r}\right). \end{aligned}$$

应用有限元插值公式, 变形张量有限元插值为

$$\left\{ \begin{aligned} e_{\alpha\beta}(\mathbf{w}_h) &= e_{\alpha\beta}(\mathbf{w}(i))H_i(\xi), \\ e_{3\alpha}(\mathbf{w}_h) &= e_{3\alpha}^*(\mathbf{w}(i))H_i(\xi) + \widehat{e}_{3\alpha}(\mathbf{w}(i))H_i'(\xi), \\ e_{33}(\mathbf{w}_h) &= e_{33}^*(\mathbf{w}(i))H_i(\xi) + \widehat{e}_{33}(\mathbf{w}(i))H_i'(\xi), \\ \widehat{e}_{3\alpha}(\mathbf{w}(i)) &= \frac{1}{2}[a_{\alpha\sigma}w^{\sigma}(i) + \varepsilon r^2\Theta_{\alpha}w^3(i), \\ \widehat{e}_{33}(\mathbf{w}(i)) &= (r\varepsilon)^2w^3(i) + \varepsilon r^2\Theta_{\sigma}w^{\sigma}(i). \end{aligned} \right. \quad (3.14)$$

引入记号

$$\begin{aligned} A_{\alpha\beta}^{ij}(\mathbf{w}_k(i)) &= \alpha_{ij}\mathcal{E}_{\alpha\beta}(\mathbf{w}_k(i)) + \alpha_{ij}^{(1)}\widetilde{\mathcal{E}}_{\alpha\beta}(\mathbf{w}_k(i)), \\ A_{3\alpha}^{*,ij}(\mathbf{w}_k(i)) &= \alpha_{ij}\mathcal{E}_{3\alpha}(\mathbf{w}_k(i)) + \alpha_{ij}^{(1)}\widetilde{\mathcal{E}}_{3\alpha}(\mathbf{w}_k(i)), \end{aligned}$$

$$\begin{aligned}\hat{A}_{3\alpha}^{ij}(\mathbf{w}_k(i)) &= \alpha_{ij}^{(2)} \mathcal{E}_{3\alpha}(\mathbf{w}_k(i)) + \alpha_{ij}^{(3)} \tilde{\mathcal{E}}_{3\alpha}(\mathbf{w}_k(i)), \\ A_{33}^{*,ij}(\mathbf{w}_k(i)) &= \alpha_{ij} \mathcal{E}_{33}(\mathbf{w}_k(i)) + \alpha_{ij}^{(1)} \tilde{\mathcal{E}}_{33}(\mathbf{w}_k(i)), \\ \hat{A}_{33}^{ij}(\mathbf{w}_k(i)) &= \alpha_{ij}^{(2)} \mathcal{E}_{33}(\mathbf{w}_k(i)) + \alpha_{ij}^{(3)} \mathcal{E}_{33}(\mathbf{w}_k(i)),\end{aligned}$$

其中

$$\left\{ \begin{aligned} \mathcal{E}_{\alpha\beta}(\mathbf{w}) &= e_{\alpha\beta}(\mathbf{w}) + (r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})e_{33}^*(\mathbf{w}) - 2\varepsilon^{-1}\Theta_\beta e_{3\alpha}^*(\mathbf{w}), \\ \tilde{\mathcal{E}}_{\alpha\beta}(\mathbf{w}) &= (r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})\hat{e}_{33}(\mathbf{w}) - 2\varepsilon^{-1}\Theta_\beta \hat{e}_{3\alpha}(\mathbf{w}), \\ \mathcal{E}_{3\alpha}(\mathbf{w}) &= -2\varepsilon^{-1}\Theta_\beta e_{\alpha\beta}(\mathbf{w}) + 2(r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})e_{3\beta}^*(\mathbf{w}) - 2\varepsilon^{-1}(r\varepsilon)^{-2}a\Theta_\alpha e_{33}^*(\mathbf{w}), \\ \tilde{\mathcal{E}}_{3\alpha}(\mathbf{w}) &= 2(r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})\hat{e}_{3\beta}(\mathbf{w}) - 2\varepsilon^{-1}(r\varepsilon)^{-2}a\Theta_\alpha \hat{e}_{33}(\mathbf{w}), \\ \mathcal{E}_{33}(\mathbf{w}) &= (r\varepsilon)^{-2}(a_{\alpha\beta} - \delta_{\alpha\beta})e_{\alpha\beta}(\mathbf{w}) - 2\varepsilon^{-1}(r\varepsilon)^{-2}a\Theta_\alpha e_{3\alpha}^*(\mathbf{w}) + (r\varepsilon)^{-4}a^2 e_{33}^*(\mathbf{w}), \\ \tilde{\mathcal{E}}_{33}(\mathbf{w}) &= -2\varepsilon^{-1}(r\varepsilon)^{-2}a\Theta_\alpha \hat{e}_{3\alpha}(\mathbf{w}) + (r\varepsilon)^{-4}a^2 \hat{e}_{33}(\mathbf{w}), \end{aligned} \right.$$

那么有下面的命题.

命题 5 在 R - 坐标系下, 在形状函数 (2.7) 所建立的有限元子空间 $S_h[-1, 1] \subset H^2([-1, 1])$ 上, 双线性形式 $a(\mathbf{w}, \mathbf{v})$ 的逼近形式为

$$\begin{aligned}a(\mathbf{w}_h, \mathbf{v}_h) &= a_0(\mathbf{u}_0, \mathbf{v}'_0) + a_m(\mathbf{u}_m, \mathbf{v}'_m) + \sum_{k=1}^{m-1} [a_k(\mathbf{w}_k, \mathbf{u}_k; \mathbf{v}_k) + a'_k(\mathbf{w}_k, \mathbf{u}_k; \mathbf{v}'_k)] \\ &\quad + \tilde{a}_0(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}'_0) + \tilde{a}_m(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, \mathbf{v}'_m) \\ &\quad + \sum_{k=1}^{m-1} [\tilde{a}_k(\mathbf{w}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_k) + \tilde{a}'_k(\mathbf{w}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}'_k)],\end{aligned}$$

其中

$$\begin{aligned}a_0(\mathbf{u}_0, \mathbf{v}'_0) &= 2\mu[(A_{\alpha\beta}^{22}(\mathbf{u}_0), e_{\alpha\beta}(\mathbf{v}'_0)) + (A_{3\alpha}^{*,22}(\mathbf{u}_0), e_{3\alpha}^*(\mathbf{v}'_0)) + (\hat{A}_{3\alpha}^{22}(\mathbf{u}_0), \hat{e}_{3\alpha}(\mathbf{v}'_0)) \\ &\quad + (A_{33}^{*,22}(\mathbf{u}_0), e_{33}^*(\mathbf{v}'_0)) + (\hat{A}_{33}^{22}(\mathbf{u}_0), \hat{e}_{33}(\mathbf{v}'_0))], \\ a_m(\mathbf{u}_m, \mathbf{v}'_m) &= 2\mu[(A_{\alpha\beta}^{44}(\mathbf{u}_m), e_{\alpha\beta}(\mathbf{v}'_m)) + (A_{3\alpha}^{*,44}(\mathbf{u}_m), e_{3\alpha}^*(\mathbf{v}'_m)) + (\hat{A}_{3\alpha}^{44}(\mathbf{u}_m), \hat{e}_{3\alpha}(\mathbf{v}'_m)) \\ &\quad + (A_{33}^{*,44}(\mathbf{u}_m), e_{33}^*(\mathbf{v}'_m)) + (\hat{A}_{33}^{44}(\mathbf{u}_m), \hat{e}_{33}(\mathbf{v}'_m))], \\ \tilde{a}_0(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}'_0) &= 2\mu[(A_{\alpha\beta}^{32}(\mathbf{w}_1) + A_{\alpha\beta}^{42}(\mathbf{u}_1), e_{\alpha\beta}(\mathbf{v}'_0)) + (A_{3\alpha}^{*,32}(\mathbf{w}_1) + A_{3\alpha}^{*,42}(\mathbf{u}_1), e_{3\alpha}^*(\mathbf{v}'_0)) \\ &\quad + (\hat{A}_{3\alpha}^{32}(\mathbf{w}_1) + \hat{A}_{3\alpha}^{42}(\mathbf{u}_1), \hat{e}_{3\alpha}(\mathbf{v}'_0)) + (A_{33}^{*,32}(\mathbf{w}_1) + A_{33}^{*,42}(\mathbf{u}_1), e_{33}^*(\mathbf{v}'_0)) \\ &\quad + (\hat{A}_{33}^{32}(\mathbf{w}_1) + \hat{A}_{33}^{42}(\mathbf{u}_1), \hat{e}_{33}(\mathbf{v}'_0))], \\ \tilde{a}_m(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, \mathbf{v}'_m) &= 2\mu[(A_{\alpha\beta}^{14}(\mathbf{w}_{m-1}) + A_{\alpha\beta}^{24}(\mathbf{u}_{m-1}), e_{\alpha\beta}(\mathbf{v}'_m)) \\ &\quad + (A_{3\alpha}^{*,14}(\mathbf{w}_{m-1}) + A_{3\alpha}^{*,24}(\mathbf{u}_{m-1}), e_{3\alpha}^*(\mathbf{v}'_m)) \\ &\quad + (\hat{A}_{3\alpha}^{14}(\mathbf{w}_{m-1}) + \hat{A}_{3\alpha}^{24}(\mathbf{u}_{m-1}), \hat{e}_{3\alpha}(\mathbf{v}'_m)) \\ &\quad + (A_{33}^{*,14}(\mathbf{w}_{m-1}) + A_{33}^{*,24}(\mathbf{u}_{m-1}), e_{33}^*(\mathbf{v}'_m)) \\ &\quad + (\hat{A}_{33}^{14}(\mathbf{w}_{m-1}) + \hat{A}_{33}^{24}(\mathbf{u}_{m-1}), \hat{e}_{33}(\mathbf{v}'_m))], \\ a_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) &= (A_{\alpha\beta}^{33}(\mathbf{w}_k) + A_{\alpha\beta}^{11}(\mathbf{u}_k) + A_{\alpha\beta}^{43}(\mathbf{u}_k) + A_{\alpha\beta}^{21}(\mathbf{u}_k), e_{\alpha\beta}(\mathbf{v}_k))\end{aligned}$$

$$\begin{aligned}
& + (A_{3\alpha}^{*,33}(\mathbf{w}_k) + A_{3\alpha}^{*,11}(\mathbf{w}_k) + A_{3\alpha}^{*,43}(\mathbf{u}_k) + A_{3\alpha}^{*,21}(\mathbf{u}_k), e_{3\alpha}^*(\mathbf{v}_k)) \\
& + (\hat{A}_{3\alpha}^{33}(\mathbf{w}_k) + \hat{A}_{3\alpha}^{11}(\mathbf{w}_k) + \hat{A}_{3\alpha}^{43}(\mathbf{u}_k) + \hat{A}_{3\alpha}^{21}(\mathbf{u}_k), \hat{e}_{3\alpha}(\mathbf{v}_k)) \\
& + (A_{33}^{*,33}(\mathbf{w}_k) + A_{33}^{*,11}(\mathbf{w}_k) + A_{43}^{*,43}(\mathbf{u}_k) + A_{33}^{*,21}(\mathbf{u}_k), e_{33}^*(\mathbf{v}_k)) \\
& + (\hat{A}_{33}^{33}(\mathbf{w}_k) + \hat{A}_{33}^{11}(\mathbf{w}_k) + \hat{A}_{33}^{43}(\mathbf{u}_k) + \hat{A}_{33}^{21}(\mathbf{u}_k), \hat{e}_{33}(\mathbf{v}_k)), \\
a'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) & = (A_{\alpha\beta}^{34}(\mathbf{w}_k) + A_{\alpha\beta}^{12}(\mathbf{w}_k) + A_{\alpha\beta}^{44}(\mathbf{u}_k) + A_{\alpha\beta}^{22}(\mathbf{u}_k), e_{\alpha\beta}(\mathbf{v}'_k)) \\
& + (A_{3\alpha}^{*,34}(\mathbf{w}_k) + A_{3\alpha}^{*,12}(\mathbf{w}_k) + A_{3\alpha}^{*,44}(\mathbf{u}_k) + A_{3\alpha}^{*,22}(\mathbf{u}_k), e_{3\alpha}^*(\mathbf{v}'_k)) \\
& + (\hat{A}_{3\alpha}^{34}(\mathbf{w}_k) + \hat{A}_{3\alpha}^{12}(\mathbf{w}_k) + \hat{A}_{3\alpha}^{44}(\mathbf{u}_k) + \hat{A}_{3\alpha}^{22}(\mathbf{u}_k), \hat{e}_{3\alpha}(\mathbf{v}'_k)) \\
& + (A_{33}^{*,34}(\mathbf{w}_k) + A_{33}^{*,12}(\mathbf{w}_k) + A_{33}^{*,44}(\mathbf{u}_k) + A_{33}^{*,22}(\mathbf{u}_k), e_{33}^*(\mathbf{v}'_k)) \\
& + (\hat{A}_{33}^{34}(\mathbf{w}_k) + \hat{A}_{33}^{12}(\mathbf{w}_k) + \hat{A}_{33}^{44}(\mathbf{u}_k) + \hat{A}_{33}^{22}(\mathbf{u}_k), \hat{e}_{33}(\mathbf{v}'_k)), \\
\tilde{a}_k(\mathbf{w}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}_k) & \\
& = (A_{\alpha\beta}^{13}(\mathbf{w}_{k-1}) + A_{\alpha\beta}^{23}(\mathbf{u}_{k-1}) + A_{\alpha\beta}^{31}(\mathbf{w}_{k+1}) + A_{\alpha\beta}^{43}(\mathbf{u}_{k+1}), e_{\alpha\beta}(\mathbf{v}_k)) \\
& + (A_{3\alpha}^{*,13}(\mathbf{w}_{k-1}) + A_{3\alpha}^{*,23}(\mathbf{u}_{k-1}) + A_{3\alpha}^{*,31}(\mathbf{w}_{k+1}) + A_{3\alpha}^{*,43}(\mathbf{u}_{k+1}), e_{3\alpha}^*(\mathbf{v}_k)) \\
& + (\hat{A}_{3\alpha}^{13}(\mathbf{w}_{k-1}) + \hat{A}_{3\alpha}^{23}(\mathbf{u}_{k-1}) + \hat{A}_{3\alpha}^{31}(\mathbf{w}_{k+1}) + \hat{A}_{3\alpha}^{43}(\mathbf{u}_{k+1}), \hat{e}_{3\alpha}(\mathbf{v}_k)) \\
& + (A_{33}^{*,13}(\mathbf{w}_{k-1}) + A_{33}^{*,23}(\mathbf{u}_{k-1}) + A_{33}^{*,31}(\mathbf{w}_{k+1}) + A_{33}^{*,43}(\mathbf{u}_{k+1}), e_{33}^*(\mathbf{v}_k)) \\
& + (\hat{A}_{33}^{13}(\mathbf{w}_{k-1}) + \hat{A}_{33}^{23}(\mathbf{u}_{k-1}) + \hat{A}_{33}^{31}(\mathbf{w}_{k+1}) + \hat{A}_{33}^{43}(\mathbf{u}_{k+1}), \hat{e}_{33}(\mathbf{v}_k)), \\
\tilde{a}'_k(\mathbf{w}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k+1}, \mathbf{u}_{k+1}, \mathbf{v}'_k) & \\
& = (A_{\alpha\beta}^{14}(\mathbf{w}_{k-1}(i)) + A_{\alpha\beta}^{24}(\mathbf{u}_{k-1}(i)) + A_{\alpha\beta}^{32}(\mathbf{w}_{k+1}(i)) + A_{\alpha\beta}^{42}(\mathbf{u}_{k+1}(i)), e_{\alpha\beta}(\mathbf{v}'_k)) \\
& + (A_{3\alpha}^{*,14}(\mathbf{w}_{k-1}(i)) + A_{3\alpha}^{*,24}(\mathbf{u}_{k-1}(i)) + A_{3\alpha}^{*,32}(\mathbf{w}_{k+1}(i)) + A_{3\alpha}^{*,42}(\mathbf{u}_{k+1}(i)), e_{3\alpha}^*(\mathbf{v}'_k)) \\
& + (\hat{A}_{3\alpha}^{14}(\mathbf{w}_{k-1}(i)) + \hat{A}_{3\alpha}^{24}(\mathbf{u}_{k-1}(i)) + \hat{A}_{3\alpha}^{32}(\mathbf{w}_{k+1}(i)) + \hat{A}_{3\alpha}^{42}(\mathbf{u}_{k+1}(i)), \hat{e}_{3\alpha}(\mathbf{v}'_k)) \\
& + (A_{33}^{*,14}(\mathbf{w}_{k-1}(i)) + A_{33}^{*,24}(\mathbf{u}_{k-1}(i)) + A_{33}^{*,32}(\mathbf{w}_{k+1}(i)) + A_{33}^{*,42}(\mathbf{u}_{k+1}(i)), e_{33}^*(\mathbf{v}'_k)) \\
& + (\hat{A}_{33}^{14}(\mathbf{w}_{k-1}(i)) + \hat{A}_{33}^{24}(\mathbf{u}_{k-1}(i)) + \hat{A}_{33}^{32}(\mathbf{w}_{k+1}(i)) + \hat{A}_{33}^{42}(\mathbf{u}_{k+1}(i)), \hat{e}_{33}(\mathbf{v}'_k)).
\end{aligned}$$

命题 6 在命题 5 的假定下, 三线性形式 $b(\mathbf{w}, \mathbf{w}, \mathbf{v})$ 和 Coriolis 力 (4.1) 的有项元逼近形式为

$$\begin{aligned}
b_h(\mathbf{w}_h, \mathbf{w}_h, \mathbf{v}_h) & = b_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}'_0) + b_m(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}'_m) + \tilde{b}_0(\mathbf{W}_1, \mathbf{W}_1, \mathbf{v}'_0) + \tilde{b}_m(\mathbf{W}_{m-1}, \mathbf{W}_{m-1}, \mathbf{v}'_0) \\
& + \sum_{k=1}^{m-1} [b_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) + b'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) + \tilde{b}_k(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}, \mathbf{v}_k) \\
& + \tilde{b}'_k(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}, \mathbf{v}'_k)], \\
C(\mathbf{w}_h, \mathbf{v}_h) & = C_0(\mathbf{u}_0, \mathbf{v}'_0) + C_m(\mathbf{u}_0, \mathbf{v}'_m) + \sum_{k=1}^{m-1} [C_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) + C'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k)] + \tilde{C}_0(\mathbf{W}_1, \mathbf{v}'_0) \\
& + \tilde{C}_m(\mathbf{W}_{m-1}, \mathbf{v}'_m) + \sum_{k=1}^{m-1} [\tilde{C}_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) + \tilde{C}'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}'_k)],
\end{aligned}$$

其中

$$\begin{aligned}
\mathbf{W}_{k+1} & = \{\mathbf{w}_{k+1}, \mathbf{u}_{k+1}\}, \quad \mathbf{W}_{k-1} = \{\mathbf{w}_{k-1}, \mathbf{u}_{k-1}\}, \\
b_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}'_0) & = (\alpha_{22,2} \mathcal{B}_i(\mathbf{u}_0, \mathbf{u}_0) + \alpha_{22,2}^{(1)} \tilde{\mathcal{B}}_i(\mathbf{u}_0, \mathbf{u}_0), v_0^i),
\end{aligned}$$

$$\begin{aligned}
 b_m(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}'_m) &= (\alpha_{44,4} \mathcal{B}_i(\mathbf{u}_m, \mathbf{u}_m) + \alpha_{44,4}^{(1)} \tilde{\mathcal{B}}_i(\mathbf{u}_m, \mathbf{u}_m), v'_m), \\
 b_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) &= (\mathcal{B}_i^k(\mathbf{w}_k, \mathbf{u}_k), v_k^i), \quad b'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) = (\mathcal{B}'_i^k(\mathbf{w}_k, \mathbf{u}_k), v_k'^i), \\
 \mathcal{B}_\beta(\mathbf{w}_k, \mathbf{w}_k) &= a_{\alpha\beta} \left[w_k^\lambda \frac{\partial w_k^\beta}{\partial x^\lambda} + \pi_{ij}^\beta w_k^i w_k^j \right] + r\varepsilon^2 \Theta_\alpha \left[w_k^\lambda \frac{\partial w_k^3}{\partial x^\lambda} + \pi_{ij}^3 w_k^i w_k^j \right], \\
 \tilde{\mathcal{B}}_\beta(\mathbf{w}_k, \mathbf{w}_k) &= a_{\alpha\beta} w_k^3 w_k^\alpha + r\varepsilon^2 \Theta_\beta w_k^3 w_k^3, \\
 \mathcal{B}_3(\mathbf{w}_k, \mathbf{w}_k) &= r^2 \varepsilon \Theta_\beta \left[w_k^\lambda \frac{\partial w_k^\beta}{\partial x^\lambda} + \pi_{ij}^\beta w_k^i w_k^j \right] + r^2 \varepsilon^2 \left[w_k^\lambda \frac{\partial w_k^3}{\partial x^\lambda} + \pi_{ij}^3 w_k^i w_k^j \right], \\
 \tilde{\mathcal{B}}_3(\mathbf{w}_k, \mathbf{w}_k) &= r^2 \varepsilon \Theta_\beta w_k^3 w_k^\beta + r^2 \varepsilon^2 w_k^3 w_k^3, \\
 \mathcal{B}_i^k(\mathbf{w}_k, \mathbf{u}_k) &= \mathcal{B}_i(\alpha_{11,1} \mathbf{w}_k + \alpha_{21,1} \mathbf{u}_k, \mathbf{w}_k) + \mathcal{B}_i(\alpha_{12,1} \mathbf{w}_k + \alpha_{22,1} \mathbf{u}_k, \mathbf{u}_k) \\
 &\quad + \mathcal{B}_i(\mathbf{w}_k, \alpha_{33,3} \mathbf{w}_k + \alpha_{34,3} \mathbf{u}_k) + \mathcal{B}_i(\mathbf{u}_k, \alpha_{43,3} \mathbf{w}_k + \alpha_{44,3} \mathbf{u}_k) \\
 &\quad + \tilde{\mathcal{B}}_i(\alpha_{11,1}^{(1)} \mathbf{w}_k + \alpha_{21,1}^{(1)} \mathbf{u}_k, \mathbf{w}_k) + \tilde{\mathcal{B}}_i(\alpha_{12,1}^{(1)} \mathbf{w}_k + \alpha_{22,1}^{(1)} \mathbf{u}_k, \mathbf{u}_k) \\
 &\quad + \tilde{\mathcal{B}}_i(\mathbf{w}_k, \alpha_{33,3}^{(1)} \mathbf{w}_k + \alpha_{34,3}^{(1)} \mathbf{u}_k) + \tilde{\mathcal{B}}_i(\mathbf{u}_k, \alpha_{43,3}^{(1)} \mathbf{w}_k + \alpha_{44,3}^{(1)} \mathbf{u}_k), \\
 \mathcal{B}'_i^k(\mathbf{w}_k, \mathbf{u}_k) &= \mathcal{B}_i(\alpha_{11,2} \mathbf{w}_k + \alpha_{21,2} \mathbf{u}_k, \mathbf{w}_k) + \mathcal{B}_i(\alpha_{12,2} \mathbf{w}_k + \alpha_{22,2} \mathbf{u}_k, \mathbf{u}_k) \\
 &\quad + \mathcal{B}_i(\mathbf{w}_k, \alpha_{33,4} \mathbf{w}_k + \alpha_{34,4} \mathbf{u}_k) + \mathcal{B}_i(\mathbf{u}_k, \alpha_{43,4} \mathbf{w}_k + \alpha_{44,4} \mathbf{u}_k) \\
 &\quad + \tilde{\mathcal{B}}_i(\alpha_{11,2}^{(1)} \mathbf{w}_k + \alpha_{21,2}^{(1)} \mathbf{u}_k, \mathbf{w}_k) + \tilde{\mathcal{B}}_i(\alpha_{12,2}^{(1)} \mathbf{w}_k + \alpha_{22,2}^{(1)} \mathbf{u}_k, \mathbf{u}_k) \\
 &\quad + \tilde{\mathcal{B}}_i(\mathbf{w}_k, \alpha_{33,4}^{(1)} \mathbf{w}_k + \alpha_{34,4}^{(1)} \mathbf{u}_k) + \tilde{\mathcal{B}}_i(\mathbf{u}_k, \alpha_{43,4}^{(1)} \mathbf{w}_k + \alpha_{44,4}^{(1)} \mathbf{u}_k), \\
 \tilde{b}_0(\mathbf{W}_1, \mathbf{W}_1, \mathbf{v}'_0) &= (\hat{B}_{2,i}^0(\mathbf{W}_1, \mathbf{W}_1), v_0'^i), \\
 \tilde{b}_m(\mathbf{W}_{m-1}, \mathbf{W}_{m-1}, \mathbf{v}'_0) &= (\hat{B}_{4,i}^{m-1}(\mathbf{w}_{m-1}, \mathbf{w}_{m-1}), v_m'^i), \\
 \tilde{b}_k(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}, \mathbf{v}_k) &= (\mathcal{B}_i^{*k}(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}), v_k^i), \\
 \tilde{b}'_k(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}, \mathbf{v}'_k) &= (\mathcal{B}'_i^{*k}(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}), v_k'^i), \\
 \hat{B}_{2,i}^0(\mathbf{W}_1, \mathbf{W}_1) &= \mathcal{B}_i(\alpha_{32,2} \mathbf{w}_1 + \alpha_{42,2} \mathbf{u}_1, \mathbf{u}_0) + \mathcal{B}_i(\mathbf{u}_0, \alpha_{23,2} \mathbf{w}_1 + \alpha_{24,2} \mathbf{u}_1) \\
 &\quad + \tilde{\mathcal{B}}_i(\alpha_{32,2}^{(1)} \mathbf{w}_1 + \alpha_{42,2}^{(1)} \mathbf{u}_1, \mathbf{u}_0) + \tilde{\mathcal{B}}_i(\mathbf{u}_0, \alpha_{23,2}^{(1)} \mathbf{w}_1 + \alpha_{24,2}^{(1)} \mathbf{u}_1) \\
 &\quad + \mathcal{B}_i(\mathbf{w}_1, \alpha_{33,2} \mathbf{w}_1 + \alpha_{34,2} \mathbf{u}_1) + \mathcal{B}_i(\mathbf{u}_1, \alpha_{43,2} \mathbf{w}_1 + \alpha_{44,2} \mathbf{u}_1) \\
 &\quad + \tilde{\mathcal{B}}_i(\mathbf{w}_1, \alpha_{33,2}^{(1)} \mathbf{w}_1 + \alpha_{34,2}^{(1)} \mathbf{u}_1) + \tilde{\mathcal{B}}_i(\mathbf{u}_1, \alpha_{43,2}^{(1)} \mathbf{w}_1 + \alpha_{44,2}^{(1)} \mathbf{u}_1), \\
 \hat{B}_{4,i}^{m-1}(\mathbf{W}_{m-1}, \mathbf{W}_{m-1}) &= \mathcal{B}_i(\alpha_{14,4} \mathbf{w}_{m-1} + \alpha_{24,4} \mathbf{u}_{m-1}, \mathbf{u}_m) + \mathcal{B}_i(\mathbf{u}_m, \alpha_{41,4} \mathbf{w}_{m-1} + \alpha_{42,4} \mathbf{u}_{m-1}) \\
 &\quad + \mathcal{B}_i(\mathbf{w}_{m-1}, \alpha_{11,4} \mathbf{w}_{m-1} + \alpha_{12,4} \mathbf{u}_{m-1}) + \mathcal{B}_i(\mathbf{u}_{m-1}, \alpha_{21,4} \mathbf{w}_{m-1} + \alpha_{22,4} \mathbf{u}_{m-1}) \\
 &\quad + \tilde{\mathcal{B}}_i(\alpha_{14,4}^{(1)} \mathbf{w}_{m-1} + \alpha_{24,4}^{(1)} \mathbf{u}_{m-1}, \mathbf{u}_m) + \tilde{\mathcal{B}}_i(\mathbf{u}_m, \alpha_{41,4}^{(1)} \mathbf{w}_{m-1} + \alpha_{42,4}^{(1)} \mathbf{u}_{m-1}) \\
 &\quad + \tilde{\mathcal{B}}_i(\mathbf{w}_{m-1}, \alpha_{11,4}^{(1)} \mathbf{w}_{m-1} + \alpha_{12,4}^{(1)} \mathbf{u}_{m-1}) + \tilde{\mathcal{B}}_i(\mathbf{u}_{m-1}, \alpha_{21,4}^{(1)} \mathbf{w}_{m-1} + \alpha_{22,4}^{(1)} \mathbf{u}_{m-1}), \\
 \mathcal{B}_i^{*k}(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}) &= \mathcal{B}_i(\mathbf{w}_k, \alpha_{13,1} \mathbf{w}_{k+1} + \alpha_{14,1} \mathbf{u}_{k+1} + \alpha_{31,3} \mathbf{w}_{k-1} + \alpha_{32,3} \mathbf{u}_{k-1}) \\
 &\quad + \mathcal{B}_i(\mathbf{u}_k, \alpha_{23,1} \mathbf{w}_{k+1} + \alpha_{24,1} \mathbf{u}_{k+1} + \alpha_{41,3} \mathbf{w}_{k-1} + \alpha_{42,3} \mathbf{u}_{k-1}) \\
 &\quad + \mathcal{B}_i(\mathbf{w}_{k+1}, \alpha_{3i,1} \mathbf{w}_k(i)) + \mathcal{B}_i(\mathbf{u}_{k+1}, \alpha_{4i,1} \mathbf{w}_k(i)) + \mathcal{B}_i(\mathbf{w}_{k-1}, \alpha_{1i,1} \mathbf{w}_{k-1}(i)) \\
 &\quad + \mathcal{B}_i(\mathbf{u}_{k-1}, \alpha_{2i,1} \mathbf{w}_{k-1}(i)) + \tilde{\mathcal{B}}_i(\mathbf{u}_{k-1}, \alpha_{2i,1}^{(1)} \mathbf{w}_{k-1}(i)) \\
 &\quad + \tilde{\mathcal{B}}_i(\mathbf{w}_{k+1}, \alpha_{3i,1}^{(1)} \mathbf{w}_k(i)) + \tilde{\mathcal{B}}_i(\mathbf{u}_{k+1}, \alpha_{4i,1}^{(1)} \mathbf{w}_k(i)) + \tilde{\mathcal{B}}_i(\mathbf{w}_{k-1}, \alpha_{1i,1}^{(1)} \mathbf{w}_{k-1}(i)) \\
 &\quad + \tilde{\mathcal{B}}_i(\mathbf{w}_k, \alpha_{13,1}^{(1)} \mathbf{w}_{k+1} + \alpha_{14,1}^{(1)} \mathbf{u}_{k+1} + \alpha_{31,3}^{(1)} \mathbf{w}_{k-1} + \alpha_{32,3}^{(1)} \mathbf{u}_{k-1})
 \end{aligned}$$

$$\begin{aligned}
& + \tilde{\mathcal{B}}_i(\mathbf{u}_k, \alpha_{23,1}^{(1)} \mathbf{w}_{k+1} + \alpha_{24,1}^{(1)} \mathbf{u}_{k+1} + \alpha_{41,3}^{(1)} \mathbf{w}_{k-1} + \alpha_{42,3}^{(1)} \mathbf{u}_{k-1}), \\
\mathcal{B}_i'^{*k}(\mathbf{W}_{k+1}, \mathbf{W}_{k-1}) & = \mathcal{B}_i(\mathbf{w}_k, \alpha_{13,2} \mathbf{w}_{k+1} + \alpha_{14,2} \mathbf{u}_{k+1} + \alpha_{31,4} \mathbf{w}_{k-1} + \alpha_{32,4} \mathbf{u}_{k-1}) \\
& + \mathcal{B}_i(\mathbf{u}_k, \alpha_{23,2} \mathbf{w}_{k+1} + \alpha_{24,2} \mathbf{u}_{k+1} + \alpha_{41,4} \mathbf{w}_{k-1} + \alpha_{42,4} \mathbf{u}_{k-1}) \\
& + \mathcal{B}_i(\mathbf{w}_{k+1}, \alpha_{3i,2} \mathbf{w}_k(i) + \mathcal{B}_i(\mathbf{u}_{k+1}, \alpha_{4i,2} \mathbf{w}_k(i)) \\
& + \mathcal{B}_i(\mathbf{w}_{k-1}, \alpha_{1i,4} \mathbf{w}_{k-1}(i)) + \mathcal{B}_i(\mathbf{u}_{k-1}, \alpha_{2i,4} \mathbf{w}_{k-1}(i)) \\
& + \tilde{\mathcal{B}}_i(\mathbf{w}_k, \alpha_{13,2}^{(1)} \mathbf{w}_{k+1} + \alpha_{14,2}^{(1)} \mathbf{u}_{k+1} + \alpha_{31,4}^{(1)} \mathbf{w}_{k-1} + \alpha_{32,4}^{(1)} \mathbf{u}_{k-1}) \\
& + \tilde{\mathcal{B}}_i(\mathbf{u}_k, \alpha_{23,2}^{(1)} \mathbf{w}_{k+1} + \alpha_{24,2}^{(1)} \mathbf{u}_{k+1} + \alpha_{41,4}^{(1)} \mathbf{w}_{k-1} + \alpha_{42,4}^{(1)} \mathbf{u}_{k-1}) \\
& + \tilde{\mathcal{B}}_i(\mathbf{w}_{k+1}, \alpha_{3i,2}^{(1)} \mathbf{w}_k(i) + \tilde{\mathcal{B}}_i(\mathbf{u}_{k+1}, \alpha_{4i,2}^{(1)} \mathbf{w}_k(i)) + \tilde{\mathcal{B}}_i(\mathbf{w}_{k-1}, \alpha_{1i,4}^{(1)} \mathbf{w}_{k-1}(i)) \\
& + \tilde{\mathcal{B}}_i(\mathbf{u}_{k-1}, \alpha_{2i,4}^{(1)} \mathbf{w}_{k-1}(i)),
\end{aligned}$$

$$\begin{aligned}
C_0(\mathbf{u}_0, \mathbf{v}'_0) & = (C_{ij} \alpha_{22} u_0^i, v_0^j), \quad C_m(\mathbf{u}_m, \mathbf{v}'_m) = (C_{ij} \alpha_{44} u_m^i, v_m^j), \\
C_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) & = (C_{ij}((\alpha_{33} + \alpha_{11}) w_k^i + (\alpha_{43} + \alpha_{21}) u_k^i), v_k^j), \\
C'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) & = (C_{ij}((\alpha_{14} + \alpha_{12}) w_k^i + (\alpha_{24} + \alpha_{22}) u_k^i), v_k^j), \\
\tilde{C}_0(\mathbf{W}_1, \mathbf{v}'_0) & = (C_{ij}(\alpha_{32} w_1^i + \alpha_{42} u_1^i), v_0^j), \\
\tilde{C}_m(\mathbf{W}_{m-1}, \mathbf{v}'_m) & = (C_{ij}(\alpha_{14} w_{m-1}^i + \alpha_{24} u_{m-1}^i), v_m^j), \\
\tilde{C}_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) & = (C_{ij}[\alpha_{13} w_{k-1}^i + \alpha_{23} u_{k-1}^i + \alpha_{31} w_{k+1}^i + \alpha_{41} u_{k+1}^i], v_k^j), \\
\tilde{C}'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}'_k) & = (C_{ij}[\alpha_{14} w_{k-1}^i + \alpha_{24} u_{k-1}^i + \alpha_{32} w_{k+1}^i + \alpha_{42} u_{k+1}^i], v_k^j),
\end{aligned}$$

其中 C_{ij} 是由 (3.3) 所定义的 Coriolis 矩阵,

$$C_{\alpha\beta} = 2r\omega(\Theta_\alpha \delta_{3\beta} - \Theta_\beta \delta_{2\alpha}), \quad C_{\alpha 3} = -2r\omega \varepsilon \delta_{2\alpha}, \quad C_{3\beta} = 2r\omega \varepsilon \delta_{2\beta}, \quad C_{33} = 0.$$

命题 7 在命题 6 假设下, 压力项 (连续性算子的共轭算子) 和连续性方程分别为

$$\begin{aligned}
(p_h, \operatorname{div} \mathbf{v}_h) & = D_h(p_h, \mathbf{v}_h) \\
& = D_0(p_0, \mathbf{v}'_0) + D_m(p_m, \mathbf{v}'_m) + \sum_{k=1}^{m-1} [D_k(p_k, \mathbf{v}_k) + D'_k(p_k, \mathbf{v}'_k)] + \tilde{D}_0(p_1, \mathbf{v}'_0) \\
& \quad + \tilde{D}_m(p_m, \mathbf{v}'_m) + \sum_{k=1}^{m-1} [\tilde{D}_k(p_{k-1}, p_{k+1}, \mathbf{v}_k) + \tilde{D}'_k(p_{k-1}, p_{k+1}, \mathbf{v}'_k)], \\
(\operatorname{div} \mathbf{w}_h, q_h) & = D_h^*(\mathbf{w}_h, q_h) \\
& = D_0^*(\mathbf{u}_0, q_0) + D_m^*(\mathbf{u}_m, q_m) + \sum_{k=1}^{m-1} D_k^*(\mathbf{w}_k, \mathbf{u}_k, q_k) + \tilde{D}_0^*(\mathbf{w}_1, \mathbf{u}_1, q_0) \\
& \quad + \tilde{D}_m^*(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, q_0) + \sum_{k=1}^{m-1} \tilde{D}_k^*(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, q_k),
\end{aligned}$$

其中

$$\begin{aligned}
D_0^*(\mathbf{u}_0, q_0) & = (\gamma_{21}(\partial_\beta + r^{-1} \delta_{2\beta}) u_0^\beta + \gamma_{21}^{(1)} u_0^3, q_0), \\
D_m^*(\mathbf{u}_m, q_m) & = (\gamma_{42}(\partial_\beta + r^{-1} \delta_{2\beta}) u_m^\beta + \gamma_{42}^{(1)} u_m^3, q_m),
\end{aligned}$$

$$\begin{aligned}
D_k^*(\mathbf{w}_k, q_k) &= \{((\gamma_{11} + \gamma_{23})(\partial_\beta + r^{-1}\delta_{2\beta})\mathbf{w}_k^\beta + (\gamma_{12} + \gamma_{24})(\partial_\beta + r^{-1}\delta_{2\beta})u_k^\beta, q_k) \\
&\quad + ((\gamma_{11}^{(1)} + \gamma_{23}^{(1)})w_k^3 + (\gamma_{12}^{(1)} + \gamma_{24}^{(1)})u_k^\beta, q_k)\}, \\
\tilde{D}_0^*(\mathbf{W}_1, q_0) &= (\gamma_{31}(\partial_\beta + r^{-1}\delta_{2\beta})w_1^\beta + \gamma_{31}^{(1)}w_1^3 + \gamma_{41}(\partial_\beta + r^{-1}\delta_{2\beta})u_1^\beta + \gamma_{41}^{(1)}u_1^3, q_0), \\
\tilde{D}_m^*(\mathbf{W}_{m-1}, q_m) &= (\gamma_{12}(\partial_\beta + r^{-1}\delta_{2\beta})w_{m-1}^\beta + \gamma_{12}^{(1)}u_{m-1}^3 + \gamma_{22}(\partial_\beta + r^{-1}\delta_{2\beta})u_{m-1}^\beta + \gamma_{22}^{(1)}u_{m-1}^3, q_m), \\
\tilde{D}_k^*(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, q_k) &= (\gamma_{21}(\partial_\beta + r^{-1}\delta_{2\beta})w_{k-1}^\beta + \gamma_{22}(\partial_\beta + r^{-1}\delta_{2\beta})u_{k-1}^\beta + \gamma_{21}^{(1)}w_{k-1}^3 + \gamma_{22}^{(1)}u_{k-1}^3, q_k) \\
&\quad + (\gamma_{13}(\partial_\beta + r^{-1}\delta_{2\beta})w_{k+1}^\beta + \gamma_{14}(\partial_\beta + r^{-1}\delta_{2\beta})u_{k+1}^\beta + \gamma_{13}^{(1)}w_{k+1}^3 + \gamma_{14}^{(1)}u_{k+1}^3, q_k), \\
D_0(p_0, \mathbf{v}_0') &= (\gamma_{21}p_0, (\partial_\beta + r^{-1}\delta_{2\beta})v_0'^\beta) + (\gamma_{21}^{(1)}p_0, v_0'^3), \\
D_m(p_m, \mathbf{v}_m') &= (\gamma_{42}p_m, (\partial_\beta + r^{-1}\delta_{2\beta})v_m'^\beta) + (\gamma_{42}^{(1)}p_m, v_m'^3), \\
D_k(p_k, \mathbf{v}_k) &= ((\gamma_{11} + \gamma_{32})p_k, (\partial_\beta + r^{-1}\delta_{2\beta})v_k^\beta) + ((\gamma_{11}^{(1)} + \gamma_{32}^{(1)})p_k, v_0^3), \\
D_k'(p_k, \mathbf{v}_k') &= ((\gamma_{42} + \gamma_{21})p_k, (\partial_\beta + r^{-1}\delta_{2\beta})v_k'^\beta) + ((\gamma_{42}^{(1)} + \gamma_{21}^{(1)})p_k, v_0'^3), \\
\tilde{D}_0(p_1, \mathbf{v}_0') &= (\gamma_{22}p_1, (\partial_\beta + r^{-1}\delta_{2\beta})v_0'^\beta) + (\gamma_{22}^{(1)}p_1, v_0'^3), \\
\tilde{D}_m(p_{m-1}, \mathbf{v}_m') &= (\gamma_{41}p_{m-1}, (\partial_\beta + r^{-1}\delta_{2\beta})v_m'^\beta) + (\gamma_{41}^{(1)}p_{m-1}, v_m'^3), \\
\tilde{D}_k(p_{k-1}, p_{k+1}, \mathbf{v}_k) &= (\gamma_{31}p_{k-1} + \gamma_{12}p_{k+1}, (\partial_\beta + r^{-1}\delta_{2\beta})v_k^\beta) + ((\gamma_{31}^{(1)}p_{k-1} + \gamma_{12}^{(1)}p_{k+1}), v_0^3), \\
\tilde{D}_k'(p_{k-1}, p_{k+1}, \mathbf{v}_k') &= (\gamma_{41}p_{k-1} + \gamma_{22}p_{k+1}, (\partial_\beta + r^{-1}\delta_{2\beta})v_k'^\beta) + ((\gamma_{41}^{(1)}p_{k-1} + \gamma_{22}^{(1)}p_{k+1}), v_0'^3).
\end{aligned}$$

通过一些典型的张量运算, 可以得到命题 5-7 的结论 (参见文献 [1, 15]), 并且容易推出下面的命题.

命题 8 在命题 7 的假定下, 离心力和出口的自然边界条件为

$$\begin{cases} (\mathbf{f}_c, \mathbf{v}) = \int_D \sum_{k=0}^{m-1} \frac{\delta}{2} r(\omega)^2 \left(v_k^2 + \frac{1}{3} v_k'^2 + v_{k+1}^2 - \frac{1}{3} v_{k+1}'^2 \right) r \varepsilon dx, \\ \langle \mathbf{h}, \mathbf{v} \rangle = \sum_{k=0}^{m-1} \alpha_{ij} \langle \mathbf{h}(i), \mathbf{v}_k(j) \rangle \Big|_{\Gamma_{\text{out}}}. \end{cases} \quad (3.15)$$

4 对应于边界层变分问题的边界层边值问题和二度并行算法

由命题 5-7, 容易得到下面的定理.

定理 1 在 R -曲线坐标系下, $S_h \subset H^2(-1, 1)$ 是一个由 Hermite 形状函数 (2.7) 生成的一维有限元子空间, 那么, Navier-Stokes 方程的有限元逼近解, 在 $m+1$ 个流形上, 共有 $7(m+1) - 6 = 7m+1$ 个未知函数, 满足下列 $m+1$ 个二维流形 $\mathfrak{S}_k, k=0, 1, \dots, m$ 上的 $m+1$ 个联立变分问题: 求

$$(\mathbf{w}_k, p_k) \in V(D)^{3(m=1)} \times L_2(D), \quad k=1, 2, \dots, m,$$

使得

$$\begin{cases} a_k(\mathbf{w}_k, \mathbf{v}_k) + b_k(\mathbf{w}_k, \mathbf{w}_k, \mathbf{v}_k) + C_k(\mathbf{w}_k, \mathbf{v}_k) + D_k(p_k, \mathbf{v}_k) = \langle F_k, \mathbf{v}_k \rangle, \\ \forall \mathbf{v}_k \in V(D), \quad k=0, 1, \dots, m, \\ \hat{D}(\mathbf{w}_k, q) = 0, \quad \forall q \in L^2(D). \end{cases} \quad (4.1)$$

(4.1) 是一个联立方程组, 它可以写成, 在每个流形 $\mathfrak{S}_k$ 上只求解其上的 7 个主变量 $\{\mathbf{w}, \mathbf{u}, p\}$ 的方程组, 而其余变量被包含在右端项中. 然后进行迭代, 从而可以构造结构性的二度并行算法. 事实上, (4.1) 可以写成另一形式.

在正压力面和负压力面 ($\mathfrak{S}_0, \mathfrak{S}_m$) 以及交界面 $\mathfrak{S}_k$ 上, 其相应的边界层方程的变分问题为

$$\begin{cases} a_0(\mathbf{u}_0, \mathbf{v}'_0) + b_0(\mathbf{u}_0, \mathbf{u}_0, \mathbf{v}'_0) + C_0(\mathbf{u}_0, \mathbf{v}'_0) + D_0(p_0, \mathbf{v}'_0) \\ + (R_0(\mathbf{w}_1, \mathbf{u}_1, p_1), \mathbf{v}'_0) = \langle \mathbf{F}_0, \mathbf{v}'_0 \rangle, \quad \forall \mathbf{v}'_0 \in V(D), \\ D_0^*(\mathbf{u}_0, q_0) + \tilde{D}^*(\mathbf{w}_1, \mathbf{u}_1, q_0) = 0, \quad \forall q_0 \in L^2(D), \end{cases} \quad (4.2)$$

$$\begin{cases} a_m(\mathbf{u}_m, \mathbf{v}'_m) + b_m(\mathbf{u}_m, \mathbf{u}_m, \mathbf{v}'_m) + C_m(\mathbf{u}_m, \mathbf{v}'_m) + D_m(p_m, \mathbf{v}'_m) \\ + (R_m(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, p_{m-1}), \mathbf{v}'_m) = \langle \mathbf{F}_m, \mathbf{v}'_m \rangle, \quad \forall \mathbf{v}'_m \in V(D), \\ D_m^*(\mathbf{u}_m, q_m) + \tilde{D}_m^*(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, q_m) = 0, \quad \forall q_m \in L^2(D), \end{cases} \quad (4.3)$$

$$\begin{cases} a_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) + b_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) + C_K(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}_k) + D_k(p_k, \mathbf{v}_k) \\ + (R_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, p_{k-1}, p_{k+1}), \mathbf{v}_k) = \langle \mathbf{F}_k, \mathbf{v}_k \rangle, \\ \forall \mathbf{v}_k \in V(D), \quad k = 1, \dots, m-1, \\ a'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) + b'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) + C'_k(\mathbf{w}_k, \mathbf{u}_k, \mathbf{v}'_k) + D'_k(p_k, \mathbf{v}'_k) \\ + (R'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, p_{k-1}, p_{k+1}), \mathbf{v}'_k) = \langle \mathbf{F}'_k, \mathbf{v}'_k \rangle, \\ \forall \mathbf{v}'_k \in V(D), \quad k = 1, \dots, m-1, \\ D_k^*(\mathbf{w}_k, \mathbf{u}_k, q_0) + \tilde{D}_k^*(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, p_{k-1}, p_{k+1}, q_k) = 0, \quad \forall q_k \in L^2(D), \end{cases} \quad (4.4)$$

其中

$$\begin{aligned} (R_0(\mathbf{w}_1, \mathbf{u}_1, p_1), \mathbf{v}'_0) &= \tilde{a}_0(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}'_0) + \tilde{b}_0(\mathbf{W}_1, \mathbf{W}_1, \mathbf{v}'_0) + \tilde{C}_0(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}'_0) + \tilde{D}_0(p_1, \mathbf{v}'_0), \\ (R_m(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, p_1), \mathbf{v}'_m) &= \tilde{a}_m(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, \mathbf{v}'_m) + \tilde{b}_m(\mathbf{W}_{m-1}, \mathbf{W}_{m-1}, \mathbf{v}'_m) \\ &\quad + \tilde{C}_0(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, \mathbf{v}'_0) + \tilde{D}_0(p_{m-1}, \mathbf{v}'_m), \\ (R_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, p_{k-1}, p_{k+1}), \mathbf{v}_k) &= \tilde{a}_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) + \tilde{b}_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) \\ &\quad + \tilde{C}_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) + \tilde{D}_k(p_{k-1}, p_{k+1}, \mathbf{v}_k), \\ (R'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, p_{k-1}, p_{k+1}), \mathbf{v}_k) &= \tilde{a}'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) + \tilde{b}'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) \\ &\quad + \tilde{C}'_k(\mathbf{W}_{k-1}, \mathbf{W}_{k+1}, \mathbf{v}_k) + \tilde{D}'_k(p_{k-1}, p_{k+1}, \mathbf{v}_k). \end{aligned} \quad (4.5)$$

命题 9 对应于边界变分问题 (4.2) 的边界层边值问题如下:

(I) $\mathfrak{S}_0$ - 上的边界层边值问题: 求 $(\mathbf{u}_0, p_0) \in (H_\gamma^1(D))^3 \times L^2(D)$ 使得

$$\begin{cases} -\frac{\partial}{\partial x^\alpha} \left(K_{ji}^{\alpha\beta}(0) \frac{\partial u_0^i}{\partial x^\beta} \right) + K_{ji}^{0\beta}(0) \frac{\partial u_0^i}{\partial x^\beta} + K_{ji}^{00}(0) u_0^i \\ + \alpha_{22,2} \mathcal{B}_j(\mathbf{u}_0, \mathbf{u}_0) + \alpha_{22,2}^{(1)} \tilde{\mathcal{B}}_j(\mathbf{u}_0, \mathbf{u}_0) + \alpha_{22} C_{ji} u_0^i \\ + \left[-\gamma_{21} \partial_\beta p_0 \delta_{j\beta} + \left(\frac{\gamma_{21}}{r} \delta_{2j} + \gamma_{21}^{(1)} \delta_{3j} \right) p_0 \right] = F_j^0(w_1, u_1, p_1), \quad j = 1, 2, 3, \\ \gamma_{21} \frac{1}{r} \frac{\partial_\beta(r u_0^\beta)}{\partial x^\beta} + \gamma_{21}^{(1)} u_0^3 + \gamma_{31} \frac{1}{r} \frac{\partial_\beta(r w_1^\beta)}{\partial x^\beta} + \gamma_{31}^{(1)} w_1^3 + \gamma_{41} \frac{1}{r} \frac{\partial_\beta(r u_1^\beta)}{\partial x^\beta} + \gamma_{41}^{(1)} u_1^3 = 0; \end{cases} \quad (4.6)$$

(II) $\mathfrak{S}_m$ - 上的边界层边值问题: 求 $(\mathbf{u}_m, p_m) \in (H_\gamma^1(D))^3 \times L^2(D)$ 使得

$$\left\{ \begin{aligned} & -\frac{\partial}{\partial x^\alpha} \left(K_{ji}^{\alpha\beta}(m) \frac{\partial u_m^i}{\partial x^\beta} \right) + K_{ji}^{0\beta}(m) \frac{\partial u_m^i}{\partial x^\beta} + K_{ji}^{00}(m) u_m^i + \alpha_{44,4} \mathcal{B}_j(\mathbf{u}_m, \mathbf{u}_m) + \alpha_{44,4}^{(1)} \tilde{\mathcal{B}}_j(\mathbf{u}_m, \mathbf{u}_m) \\ & + \alpha_{44} C_{ji} u_m^i + \left[-\gamma_{42} \partial_\beta p_m \delta_{i\beta} + \left(\frac{\gamma_{42}}{r} \delta_{2i} + \gamma_{42}^{(1)} \delta_{3i} \right) p_m \right] = F_j^m(w_{m-1}, u_{m-1}, p_{m-1}), \\ & j = 1, 2, 3, \\ & \gamma_{42} \frac{1}{r} \frac{\partial_\beta(r u_m^\beta)}{\partial x^\beta} + \gamma_{42}^{(1)} u_m^3 + \gamma_{12} \frac{1}{r} \frac{\partial_\beta(r w_{m-1}^\beta)}{\partial x^\beta} + \gamma_{12}^{(1)} w_{m-1}^3 + \gamma_{22} \frac{1}{r} \frac{\partial_\beta(r u_{m-1}^\beta)}{\partial x^\beta} + \gamma_{22}^{(1)} u_{m-1}^3 = 0, \end{aligned} \right. \quad (4.7)$$

其中 $\mathcal{E}_j(\cdot, \cdot)$ 和 $\tilde{\mathcal{E}}_j(\cdot, \cdot)$ 由 (3.15) 定义,

$$\begin{aligned} K_{\sigma\mu}^{\alpha\beta}(0) &= \mu \alpha_{22} (\delta_{\alpha\mu} \delta_{\beta\sigma} + a \delta_{\alpha\beta} \delta_{\sigma\mu} + a_{\alpha\beta} \delta_{\sigma\mu} + (1-a) a_{\sigma\mu} \delta_{\alpha\beta}), \\ K_{\sigma 3}^{\alpha\beta}(0) &= \mu \alpha_{22} \varepsilon r^2 (1-a) \delta_{\alpha\beta} \Theta_\sigma, \\ K_{\sigma\mu}^{0\beta}(0) &= \mu \{ \alpha_{22} [r \Theta_\beta \Theta_\sigma \delta_{2\mu} - r a \Theta_\mu \Theta_\beta \delta_{2\sigma} + (1-a) r^2 (\Theta_\mu \Theta_{\beta\sigma} - \Theta_\sigma \Theta_{\alpha\mu})] \\ & \quad - \alpha_{22}^{(1)} [(1-a) \delta_{\beta\mu} \Theta_\sigma + (1-2a) \delta_{\mu\sigma} \Theta_\beta + (a-1) r^2 \Theta_\beta \Theta_\mu \Theta_\sigma] \\ & \quad - \alpha_{22}^{(2)} [a a_{\beta\sigma} \Theta_\mu + a_{\mu\sigma} \Theta_\beta] \}, \\ K_{\sigma 3}^{0\beta}(0) &= \mu \left\{ \alpha_{22} \left[\frac{1-a}{2} \varepsilon r^2 \Theta_{\beta\sigma} - a \varepsilon r \Theta_\beta \delta_{2\sigma} \right] + \alpha_{22}^{(2)} [(a-2) \delta_{\beta\sigma} + (3-a) a_{\beta\sigma}] \right\}, \\ K_{\sigma\mu}^{00}(0) &= \mu \frac{\partial}{\partial x^\beta} \left[\alpha_{22} \left(\frac{1-a}{2} r^2 \Theta_\sigma \Theta_{\beta\mu} - r \Theta_\beta \Theta_\sigma \delta_{2\mu} \right) \right. \\ & \quad \left. + \alpha_{22}^{(1)} ((1-2a) \delta_{\sigma\mu} \Theta_\beta + (1-a) \delta_{\beta\mu} \Theta_\sigma + (a-1) r^2 \Theta_\beta \Theta_\sigma \Theta_\mu) \right] \\ & \quad + 2\mu \left\{ \alpha_{22} \left[(3a^3 + 3a + 1) r^{-2} \delta_{2\sigma} \delta_{2\mu} - a r (\delta_{2\mu} \Theta_\alpha \Theta_{\alpha\sigma} + \delta_{2\sigma} \Theta_\alpha \Theta_{\alpha\mu}) \right. \right. \\ & \quad \left. \left. + \frac{1}{2} (a-1) r^2 \Theta_{\alpha\sigma\sigma} \Theta_{\alpha\mu} \right] + \alpha_{22}^{(1)} \left[(r\varepsilon)^{-1} \delta_{2\sigma} \Theta_\mu + 3\varepsilon^{-1} (a-2) \Theta_{\sigma\mu} \right. \right. \\ & \quad \left. \left. + \varepsilon^{-1} \left(\frac{7}{2} a + 1 \right) a_{\alpha\mu} \Theta_{\alpha\sigma} - \varepsilon^{-1} r^2 \Theta_\mu \Lambda_\Theta \Theta_{\alpha\mu} \right] \right. \\ & \quad \left. + \alpha_{22}^{(2)} [2(1-a) r^{-2} \delta_{2\sigma} + (1+a-a^2) r^{-1} \Theta_\sigma - a r \Theta_\alpha \Theta_{\alpha\sigma}] \delta_{2\mu} \right\}, \\ K_{\sigma 3}^{00}(0) &= 2\mu \{ \alpha_{22}^{(1)} [(5-4a) r^{-1} \delta_{2\sigma} + (2a-3) r^2 \Theta_\alpha \Theta_{\alpha\sigma}] \\ & \quad + 2\alpha_{22}^{(2)} [2r \Theta_2 \Theta_\sigma + r^2 \Theta_\sigma \Delta \Theta + r^2 \Theta_\alpha \Theta_{\alpha\sigma}] \}, \\ K_{3\mu}^{\alpha\beta}(0) &= \mu \alpha_{22} (1-a) \varepsilon r^2 \delta_{2\mu} \Theta_\mu, \quad K_{33}^{\alpha\beta}(0) = \mu \alpha_{22} (1-a) \varepsilon^2 r^2 \delta_{\alpha\beta}, \\ K_{3\mu}^{\alpha 0}(0) &= 2\mu \left[\alpha_{22} \left(-r \varepsilon a \delta_{2\mu} \Theta_\alpha + \frac{1-a}{2} \varepsilon r^2 \Theta_{\alpha\mu} \right) + \alpha_{22}^{(1)} (\delta_{\alpha\mu} - a_{\alpha\mu}) \right], \\ K_{33}^{\alpha 0}(0) &= -\mu \alpha_{22}^{(1)} r^2 \varepsilon \Theta_\alpha, \\ K_{3\mu}^{0\beta}(0) &= 2\mu \left\{ \alpha_{22} \left(r \varepsilon a \delta_{2\mu} \Theta_\beta + \frac{a-1}{2} r^2 \varepsilon \Theta_{\beta\mu} \right) + \alpha_{22}^{(1)} (a_{\beta\mu} - \delta_{\beta\mu}) + \alpha_{22}^{(3)} \frac{1-a^2}{2} \varepsilon \delta_{\beta\mu} \right\}, \\ K_{33}^{0\beta}(0) &= \mu [\alpha_{22}^{(1)} r^2 \varepsilon \Theta_\beta - \alpha_{22}^{(3)} (a+1) (r\varepsilon)^2 \Theta_\beta], \\ K_{3\mu}^{00}(0) &= 2\mu \left\{ \alpha_{22} [(2-3a) \varepsilon \delta_{2\mu} \Theta_2 + (1-a) r \zeta \Theta_{2\mu} - a r \varepsilon \delta_{2\mu} \Delta \Theta \right. \\ & \quad \left. - 2r^3 \varepsilon \delta_{2\mu} \Theta_\lambda \Theta_\alpha \Theta_{\lambda\alpha} + (1-a) r^2 \varepsilon \Delta \Theta_\mu - 2r^4 \varepsilon \Theta_\lambda \Theta_{\lambda\alpha} \Theta_{\alpha\mu}] \right\} \end{aligned}$$

$$\begin{aligned}
& + \alpha_{22}^{(1)} \left(\frac{1}{2} (2 - a - a^2) \Theta_\mu - 2r \Theta_2 \Theta_\mu + r^2 \Delta \Xi \Theta_\mu + r^2 \Theta_\alpha \Theta_{\alpha\mu} \right) \\
& + \alpha_{22}^{(2)} \left(\varepsilon (1 - a) (2a + 1) r^{-1} - \frac{1}{2} (a + 1) \varepsilon r^2 \Theta_\alpha \Theta_{\alpha\mu} \right) \Big\}, \\
K_{33}^{00}(0) &= 2\mu \left[\alpha_{22}^{(1)} \left(\frac{1}{2} r^2 \varepsilon \Delta \Theta + r \varepsilon \Theta_2 \right) - \frac{1}{2} \alpha_{22}^{(2)} \varepsilon (a - 1)^2 \right],
\end{aligned}$$

在以上 13 个等式中用 $\alpha_{44}, \alpha_{44}^{(i)}$ 代替 $\alpha_{22}, \alpha_{22}^{(i)}$ 就可以得到关于 $K_{ij}^{\alpha\beta}(m)$ 等的计算公式. 而 $F_j^m(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, p_{m-1})$ 和 $F_j^0(\mathbf{w}_1, \mathbf{u}_1, p_1)$ 由

$$\begin{aligned}
& \tilde{a}_0(\mathbf{w}_1, \mathbf{u}_1, \mathbf{v}'_0), \quad \tilde{a}_0(\mathbf{w}_{m-1}, \mathbf{u}_{m-1}, \mathbf{v}'_m), \quad \hat{B}_{2,i}^0(\mathbf{W}_1, \mathbf{W}_1), \quad \hat{B}_{4,i}^m(\mathbf{W}_{m-1}, \mathbf{W}_{m-1}), \\
& \tilde{C}_0(\mathbf{W}_1, \mathbf{v}'_0), \quad \tilde{C}_m(\mathbf{W}_{m-1}, \mathbf{v}'_m), \quad \tilde{D}_0(p_1, \mathbf{v}'_0), \quad \tilde{D}_m(p_{m-1}, \mathbf{v}'_m)
\end{aligned}$$

所确定.

证明 要由 (4.2)–(4.4) 推出相应的边值 (4.6) 和 (4.7), 可应用二维流形上的 Gauss 定理和 Green 公式以及二维流形上的一些微分算子和其他几何量的一些公式 (参见文献 [1, 9, 10]). 详细的证明, 将在后续文章中给出, 这里省略. $\square$

二度并行算法 (4.2) 和 (4.3) 一共有 $m+1$ 个方程组, 可以同时并行求解其结果分别送到邻近生成新的方程右端. 反复迭代直到收敛. 同时, 在解每个方程组时, 也可以用并行算法, 这就达到两个方向同时并行的效果, 有关收敛性的讨论可参见文献 [15–20].

5 数值计算例子

我们取 NASA 的基准例题, 分别用 CFD 的软件 FLUENT 和我们的 BLE 计算比较. 计算表明, 我们的方法是成功的. 叶轮是半开的, 流量为 30 kg/s, 角速度为 1862 r/min. FLUENT 的网格数是 30000 (见图 5), BLE 用四边形网格 (见图 6) 元素个数是 5600.

观察在子午面上 FLUENT 计算的压力图 (见图 7 和 8) 和在 $\mathfrak{S}_k$ 上用 BLE 计算的压力图 (见图 7 和 8), 结果表明, FLUENT 的结果可以改善, 而 BLE 的结果很好, 可以用于叶形的优化设计.

图 5 FLUENT 的网格

图 6 BLE 的网格

图 7 正压面的压力分布

图 8 负压面的压力分布

下面给出 BLE 计算的叶轮功率 (参见文献 [1-3])

$$W_{\text{impeller}} = \int_{\mathfrak{S}_0 \cup \mathfrak{S}_m} \sigma_{i,j} n^j r \omega (\mathbf{e}_\theta)^i d\mathbf{x},$$

应力张量 $\sigma_{ij} = -pg_{ij} + \nu(g_{jl}\nabla_i u^l + g_{il}\nabla_j u^l)$, 叶片的单位法向量 $n^\alpha = -x^2\theta_\alpha/a^{1/2}$, $n^3 = (1 + r^2\theta_2^2)/(\epsilon r a^{1/2})$, 而面元 $d\mathbf{x} = a^{1/2}dx^1dx^2$ (参见文献 [1-3]),

$$\begin{aligned} W_{\text{impeller}} = & \int_D [-(1 - r^2\Theta_1^2)p_0 + \mu\epsilon^{-1}\Theta_\alpha(1 + r^2(\Theta_1^2 - \Theta_1^2))u_0^\alpha] r \omega dx^1 dx^2 \\ & + \int_D [-(1 - r^2\Theta_1^2)p_m + \mu\epsilon^{-1}\Theta_\alpha(1 + r^2(\Theta_1^2 - \Theta_1^2))u_m^\alpha] r \omega dx^1 dx^2, \end{aligned}$$

计算表明,

$$W_{\text{impeller}} = 1,3962w \quad (\text{由 BLE 计算}), \quad W_{\text{impeller}} = 1,3973w \quad (\text{由 FLUENT 计算}).$$

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A finite element dimensional splitting method for 3D rotating Navier-Stokes equations

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Abstract In this paper, a finite element splitting method for rotating Navier-Stokes equations is established. It is well known that the channel in turbo-machinery is bounded by two adjacent blades, top surface (shroud), bottom surface (hub), inlet and outlet. The channel is divided into several sub-channels by a series of the surfaces which are isomorphic to the blade's surface. Based on a special curvilinear coordinate system, the velocity is approximated by the Hermite finite element with 3 degree and the pressure is approximated by the Lagrange finite element with 1 degree on the rotating direction. The Navier-Stokes equations are expressed into the sum of membrane and bending operators. Applying finite element expansion for the velocity and pressure leads to the finite element algebraic equations whose unknown variable are the approximated function value of the velocity and pressure and derivative value of the velocity with respect to transverse variable at the nodes. Each surface represents a two dimensional manifold in the channel. There are 8 unknown functions on each two dimensional

manifold and total unknown functions satisfy a system of partial differential equations. In particular, they are called boundary layer equations on positive blade surface and negative blade surface. The system can be solved by a two scale parallel algorithm which is proposed in this paper.

Keywords rotating Navier-Stokes equations, finite element dimensional splitting method, boundary layer equations

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