完备 Kähler 流形上 Kähler-Ricci 流的局部 Harnack 估计

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摘要 本文首先给出非正规化 Kähler-Ricci 流下曲率的发展方程, 然后得到了关于曲率的 Harnack 量在满足曲率局部条件下所产生的一个特殊项 CNS. 通过对 CNS 的估计, 得到了完备 Kähler 流形上关于 Kähler-Ricci 流的局部 Harnack 不等式. 最后, 作为主要定理的应用, 我们将结果推广到数量曲率的情形.

关键词 局部 Harnack 估计 Kähler-Ricci 流

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1 引言

几何流的 Harnack 估计也称为 Li-Yau-Hamilton 不等式. Hamilton 对 Ricci 流和平均曲率流情形的 Harnack 估计做了重要的工作, 他得到了满足曲率条件

$$-\frac{m(t)}{2}(g_{ik}g_{jl} - g_{il}g_{kj}) \leq R_{ijkl} \leq \frac{M}{2}(g_{ij}g_{kl} - g_{il}g_{kj})$$

时关于 Ricci 流的局部 Harnack 估计. Wang 和 Fang^[1,2] 用类似的方法得到了平均曲率流和 Yamabe 流下的局部 Harnack 估计. 在 [3] 中, Cao 证明了具有正的全纯双截面曲率的 Kähler-Ricci 流下的 Harnack 不等式. 本文采用 Hamilton 的方法给出了一个完备 Kähler 流形上 Kähler-Ricci 流的局部 Harnack 估计.

我们首先研究 n 维完备 Kähler 流形 X 上的 Kähler-Ricci 流

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}, \quad (1.1)$$

这里用 $R_{i\bar{j}}$ 表示关于 $g_{i\bar{j}}$ 的 Ricci 曲率.

设 $r > 0$, 令 $o \in X^n$, $B_r(o, t)$ 表示 Kähler 流形 X 上球心在 o 、 t 时刻半径为 r 的测地球, 这里用 $d = d_t(x, o)$ 表示在度量 $g_{i\bar{j}}(t)$ 下从 o 到 x 的测地距离函数.

我们有如下的主要定理:

引用格式: Zhu X R. Local Harnack estimate for Kähler-Ricci flow on complete Kähler manifold (in Chinese). Sci Sin Math, 2010, 40(9): 873–879, doi: 10.1360/012009-988

定理 1.1 设 $(g_{i\bar{j}}(x, t), X)$ 是一个满足方程 (1.1) 的 n 维 Kähler 流形, 假设 $M \geq 1$, $Mr^2 = 1$, 且在 $B_{r^2}(o, e^{n+1}r) \times [0, r^2]$ 处有

$$-\frac{m(t)}{2}G_{i\bar{j}k\bar{l}} \leq R_{i\bar{j}k\bar{l}} \leq \frac{M}{2}G_{i\bar{j}k\bar{l}},$$

$m(t)$ 在 $[0, r^2]$ 上满足方程 $m'(t) = AmM$ 及 $0 \leq m(0) \leq e^{-A}M$, 则对任意的 $(x, t) \in B_{r^2}(o, \frac{1}{2}e^{-\frac{n+1}{4}}r) \times (0, r^2]$ 和 $w \neq 0 \in T_x X$, 存在某个足够大的常数 $B > 0$, 使得如下估计成立: 设

$$\begin{aligned} \widetilde{Q}_{i\bar{j}} &= \Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{l}}R_{l\bar{k}} + R_{i\bar{j},k}v^k + R_{i\bar{j},\bar{k}}v^{\bar{k}} + \left(R_{i\bar{j}k\bar{l}} + \frac{m}{2}G_{i\bar{j}k\bar{l}}\right)v^kv^{\bar{l}} \\ &\quad + \frac{m}{2}(Rg_{i\bar{j}} + R_{i\bar{j}}) + (n+1)BM^2\left(\frac{1}{(r-d)^2} + \frac{1}{t}\right)g_{i\bar{j}}, \end{aligned}$$

我们有

$$\widetilde{Q}_{i\bar{j}}w^iw^{\bar{j}} \geq 0,$$

这里 $G_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}$, 常数 A 将在性质 2.1 的证明中给出, 这里 $A > 0$.

注记 1.1 $m'(t) = AmM$ 及 $0 \leq m(0) \leq e^{-A}M$ 保证了对所有的 $t \in [0, r^2]$, 有 $0 \leq m(t) \leq M$.

注记 1.2 由

$$-\frac{m(t)}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \leq R_{i\bar{j}k\bar{l}} \leq \frac{M}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$$

可知,

$$e^{-\frac{n+1}{2}(1-Mt)}g(t) \leq g(r^2) \leq e^{\frac{n+1}{2}(1-Mt)}g(t),$$

则

$$e^{-\frac{n+1}{4}}dt \leq d_{r^2} \leq e^{\frac{n+1}{4}}dt.$$

而 $B_{r^2}(o, e^{-\frac{n+1}{4}}r) \subset B_t(o, r) \subset B_{r^2}(o, e^{\frac{n+1}{4}}r)$, $B_{r^2}(o, \frac{1}{2}e^{-\frac{n+1}{4}}r) \subset B_t(o, \frac{1}{2}r) \subset B_{r^2}(o, \frac{1}{2}e^{\frac{n+1}{4}}r)$, 所以我们可以利用 Shi 在 [4] 中的结果, 易知在 $B_{r^2}(o, \frac{1}{2}e^{-\frac{n+1}{4}}r) \times (0, r^2]$ 上, 曲率的各阶导数一致有界.

根据定理 1.1, 我们有如下推论:

推论 1.1 若 $(g_{i\bar{j}}(x, t), X)$ 是方程 (1.1) 的解, 则在 o 点处数量曲率 R 满足估计:

$$\begin{aligned} \frac{\partial R}{\partial t} + R_{,k}v^k + R_{,\bar{k}}v^{\bar{k}} + \left[R_{i\bar{j}} + \frac{m}{2}(n+1)g_{i\bar{j}}\right]v^iv^{\bar{j}} + \frac{m(n+1)}{2}R \\ + BM^2n(n+1)\left(\frac{1}{(r-d)^2} + \frac{1}{t}\right) \geq 0. \end{aligned}$$

推论 1.2 若 $(g_{i\bar{j}}(x, t), X)$ 是方程 (1.1) 的解, 则在 o 点处有如下的估计:

(i) $|R_{,k}v^k + R_{,\bar{k}}v^{\bar{k}}|^2 \leq CM^3[R_{i\bar{j}} + \frac{m}{2}(n+1)g_{i\bar{j}}]v^iv^{\bar{j}};$

(ii) $R_{,k}v^k + R_{,\bar{k}}v^{\bar{k}} = 0$, 若 $[R_{i\bar{j}} + \frac{m}{2}(n+1)g_{i\bar{j}}]v^iv^{\bar{j}} = 0$ 或 $\frac{\partial R}{\partial t} + \frac{m(n+1)}{2}R + BM^2n(n+1)(\frac{1}{r^2} + \frac{1}{t}) = 0$.

其中 C 仅与 n 有关.

2 关于特殊项 CNS 的估计

本节首先给出 Kähler-Ricci 流下曲率的发展方程. 设 $(X, g_{i\bar{j}}(x, t))$ 是方程 (1.1) 的解, 则有以下的曲率发展方程成立:

$$\frac{\partial R_{i\bar{j}k\bar{l}}}{\partial t} = \Delta R_{i\bar{j}k\bar{l}} + R_{i\bar{j}p\bar{q}}R_{q\bar{p}k\bar{l}} - R_{i\bar{p}k\bar{q}}R_{p\bar{j}q\bar{l}} + R_{i\bar{l}p\bar{q}}R_{q\bar{p}k\bar{j}}$$

$$\begin{aligned}
& -\frac{1}{2}(R_{i\bar{p}}R_{p\bar{j}k\bar{l}} + R_{p\bar{j}}R_{i\bar{p}k\bar{l}} + R_{k\bar{p}}R_{i\bar{j}p\bar{l}} + R_{p\bar{l}}R_{i\bar{j}k\bar{p}}), \\
\frac{\partial R_{i\bar{j}}}{\partial t} &= \Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{l}}R_{l\bar{k}} - R_{i\bar{k}}R_{k\bar{j}}, \\
\frac{\partial R}{\partial t} &= \Delta R + |\text{Ric}|^2.
\end{aligned}$$

发展方程的具体计算细节可参考 [3].

定义 2.1 记

$$\begin{aligned}
\tilde{R}_{i\bar{j}k\bar{l}} &= R_{i\bar{j}k\bar{l}} + \frac{m(t)}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}), \\
Q_{i\bar{j}} &= \Delta R_{i\bar{j}} + R_{i\bar{j}k\bar{l}}R_{l\bar{k}} + R_{i\bar{j},k}v^k + R_{i\bar{j},\bar{k}}v^{\bar{k}} + R_{i\bar{j}k\bar{l}}v^kv^{\bar{l}}, \\
P_{i\bar{j}k} &= R_{i\bar{j},k} + R_{i\bar{j}kl}v^{\bar{l}}, \quad P_{i\bar{j}\bar{k}} = R_{i\bar{j},\bar{k}} + R_{i\bar{j}k\bar{l}}v^k, \\
W_{i\bar{j}k\bar{l}} &= \frac{1}{2}(R_{i\bar{j},k\bar{l}} + R_{i\bar{j},l\bar{k}}) + \frac{1}{2}R_{i\bar{j}k\bar{l},p}v^p + \frac{1}{2}R_{i\bar{j}k\bar{l},\bar{p}}v^{\bar{p}} + \frac{1}{2}(R_{i\bar{j}p\bar{l}}R_{k\bar{p}} + R_{i\bar{j}k\bar{p}}R_{p\bar{l}}), \\
Y_{l,\bar{k}} &= v_{l,\bar{k}} - R_{l\bar{k}}, \quad U^k = \left(\frac{\partial}{\partial t} - \Delta\right)v^k - \frac{1}{2}R_{p\bar{q}}v^pv^{\bar{q}}.
\end{aligned}$$

接下来我们将做一些直接的计算.

定理 2.1 设 $(g_{i\bar{j}}(x, t), X)$ 是方程 (1.1) 的解, $v \in T_x X$, 则有

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)Q_{i\bar{j}} &= R_{i\bar{j}k\bar{l}}Q_{l\bar{k}} + P_{i\bar{l}k}P_{l\bar{j}\bar{k}} + P_{i\bar{l}\bar{k}}P_{l\bar{j}k} - \frac{1}{2}(R_{i\bar{k}}Q_{k\bar{j}} + R_{k\bar{j}}Q_{i\bar{k}}) \\
&\quad - 2W_{i\bar{j}k\bar{l}}Y_{l\bar{k}} - R_{i\bar{j}k\bar{l}}Y_{l\bar{p}}Y_{p\bar{k}} + P_{i\bar{j}k}U^k + P_{i\bar{j}\bar{k}}U^{\bar{k}} - R_{i\bar{j}k\bar{l}}v_{k,\bar{p}}v_{l,p} \\
&\quad - R_{i\bar{j},kl}v_{k,l} - R_{i\bar{j},kl}v_{\bar{k},\bar{l}} - R_{i\bar{j}k\bar{l},p}v_{k,\bar{p}}v^{\bar{l}} - R_{i\bar{j}k\bar{l},\bar{p}}v_{l,p}v^k.
\end{aligned}$$

证明 做与 [3] 中类似的计算, 得到

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)Q_{i\bar{j}} &= R_{i\bar{j}k\bar{l}}Q_{l\bar{k}} + P_{i\bar{l}k}P_{l\bar{j}\bar{k}} + P_{i\bar{l}\bar{k}}P_{l\bar{j}k} + (R_{i\bar{j},k\bar{l}} + R_{i\bar{j},l\bar{k}})(R_{l\bar{k}} - v_{l,\bar{k}}) \\
&\quad + R_{i\bar{j}k\bar{l},p}(R_{l\bar{k}} - v_{l,\bar{k}})v^p + R_{i\bar{j}k\bar{l},\bar{p}}(R_{l\bar{k}} - v_{\bar{k},l})v^{\bar{p}} - \frac{1}{2}(R_{i\bar{k}}Q_{k\bar{j}} + R_{k\bar{j}}Q_{i\bar{k}}) \\
&\quad + R_{i\bar{j}k\bar{l}}(R_{l\bar{p}}R_{p\bar{k}} - v_{\bar{k},p}v_{l,\bar{p}}) + R_{i\bar{j},k}\left(\frac{\partial}{\partial t} - \Delta\right)v^k + R_{i\bar{j},\bar{k}}\left(\frac{\partial}{\partial t} - \Delta\right)v^{\bar{k}} \\
&\quad + R_{i\bar{j}k\bar{l}}\left[v^k\left(\frac{\partial}{\partial t} - \Delta\right)v^{\bar{l}} + v^{\bar{l}}\left(\frac{\partial}{\partial t} - \Delta\right)v^k\right] - R_{i\bar{j}k\bar{l}}v_{k,\bar{p}}v_{l,p} - R_{i\bar{j},kl}v_{k,l} \\
&\quad - R_{i\bar{j},kl}v_{\bar{k},\bar{l}} - R_{i\bar{j}k\bar{l},p}v_{k,\bar{p}}v^{\bar{l}} - R_{i\bar{j}k\bar{l},\bar{p}}v_{l,p}v^k \\
&= R_{i\bar{j}k\bar{l}}Q_{l\bar{k}} + P_{i\bar{l}k}P_{l\bar{j}\bar{k}} + P_{i\bar{l}\bar{k}}P_{l\bar{j}k} - \frac{1}{2}(R_{i\bar{k}}Q_{k\bar{j}} + R_{k\bar{j}}Q_{i\bar{k}}) - 2W_{i\bar{j}k\bar{l}}Y_{l\bar{k}} \\
&\quad - R_{i\bar{j}k\bar{l}}Y_{l\bar{p}}Y_{p\bar{k}} + P_{i\bar{j}k}U^k + P_{i\bar{j}\bar{k}}U^{\bar{k}} - R_{i\bar{j}k\bar{l}}v_{k,\bar{p}}v_{l,p} - R_{i\bar{j},kl}v_{k,l} - R_{i\bar{j},kl}v_{\bar{k},\bar{l}} \\
&\quad - R_{i\bar{j}k\bar{l},p}v_{k,\bar{p}}v^{\bar{l}} - R_{i\bar{j}k\bar{l},\bar{p}}v_{l,p}v^k.
\end{aligned}$$

证毕.

为了作和 [3] 中类似的讨论, 我们对定义 2.1 中的量做相应的修改之后, 其对应的 Harnack 量的表达式会产生一个特殊项, 这里用 CNS 来表示. 我们将得到关于 CNS 的一个下界估计. 首先对定

义 2.1 中的量做如下修改:

$$\begin{aligned}\tilde{R}_{i\bar{j}} &= R_{i\bar{j}} + \frac{m(t)}{2}(n+1)g_{i\bar{j}}, \quad \tilde{R} = R + \frac{m(t)}{2}(n+1)n, \\ \tilde{P}_{ijk} &= R_{i\bar{j},k} + \tilde{R}_{i\bar{j}k\bar{l}}v^{\bar{l}}, \quad \tilde{P}_{i\bar{j}\bar{k}} = R_{i\bar{j},\bar{k}} + \tilde{R}_{i\bar{j}k\bar{l}}v^k, \\ \tilde{Q}_{i\bar{j}} &= Q_{i\bar{j}} + \frac{m}{2}G_{i\bar{j}k\bar{l}}v^kv^{\bar{l}} + \frac{m}{2}(Rg_{i\bar{j}} + R_{i\bar{j}}), \\ \tilde{W}_{i\bar{j}k\bar{l}} &= W_{i\bar{j}k\bar{l}} + \frac{1}{2}\left(mg_{i\bar{j}}R_{k\bar{l}} + \frac{m}{2}g_{i\bar{l}}R_{k\bar{j}} + \frac{m}{2}g_{k\bar{j}}R_{i\bar{l}}\right), \\ \tilde{U}^k &= \left(\frac{\partial}{\partial t} - \Delta\right)v^k - \frac{1}{2}\tilde{R}_{p\bar{q}}v^pv^{k\bar{q}}.\end{aligned}$$

对比第 3 部分中的 $Q_{i\bar{j}}$, 我们希望 $\tilde{Q}_{i\bar{j}}$ 有如下形式:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\tilde{Q}_{i\bar{j}}w^iw^{\bar{j}} &= \left[R_{i\bar{j}k\bar{l}}\tilde{Q}_{l\bar{k}} + \tilde{P}_{i\bar{l}k}\tilde{P}_{l\bar{j}\bar{k}} + \tilde{P}_{i\bar{l}\bar{k}}\tilde{P}_{l\bar{j}k} - \frac{1}{2}(R_{i\bar{k}}\tilde{Q}_{k\bar{j}} + R_{k\bar{j}}\tilde{Q}_{i\bar{k}})\right. \\ &\quad - 2\tilde{W}_{i\bar{j}k\bar{l}}Y_{l\bar{k}} - \tilde{R}_{i\bar{j}k\bar{l}}Y_{l\bar{p}}Y_{p\bar{k}} + \tilde{P}_{ijk}U^{\bar{k}} + \tilde{P}_{i\bar{j}\bar{k}}U^k - \tilde{R}_{i\bar{j}k\bar{l}}v_{k,\bar{p}}v_{l,p} \\ &\quad \left.- R_{i\bar{j},k\bar{l}}v_{k,l} - R_{i\bar{j},k\bar{l}}v_{k,\bar{l}} - R_{i\bar{j}k\bar{l},p}v_{k,\bar{p}}v^{\bar{l}} - R_{i\bar{j}k\bar{l},\bar{p}}v_{l,p}v^k\right]w^iw^{\bar{j}} + \text{CNS}.\end{aligned}$$

因此不难得到

$$\begin{aligned}\text{CNS} &= \left[\left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{Q}_{i\bar{j}} - Q_{i\bar{j}}) + (R_{i\bar{j}k\bar{l}}Q_{l\bar{k}} - \tilde{R}_{i\bar{j}k\bar{l}}\tilde{Q}_{l\bar{k}}) + (P_{i\bar{l}k}P_{l\bar{j}\bar{k}} + P_{i\bar{l}\bar{k}}P_{l\bar{j}k})\right. \\ &\quad - (\tilde{P}_{i\bar{l}k}\tilde{P}_{l\bar{j}\bar{k}} + \tilde{P}_{i\bar{l}\bar{k}}\tilde{P}_{l\bar{j}k}) + \frac{1}{2}(R_{i\bar{k}}\tilde{Q}_{k\bar{j}} + R_{k\bar{j}}\tilde{Q}_{i\bar{k}}) \\ &\quad - \frac{1}{2}(R_{i\bar{k}}Q_{k\bar{j}} + R_{k\bar{j}}Q_{i\bar{k}}) + 2(\tilde{W}_{i\bar{j}k\bar{l}} - W_{i\bar{j}k\bar{l}})Y_{l\bar{k}} + (\tilde{R}_{i\bar{j}k\bar{l}} - R_{i\bar{j}k\bar{l}})Y_{l\bar{p}}Y_{p\bar{k}} \\ &\quad \left.+ P_{ijk}U^{\bar{k}} + P_{i\bar{j}\bar{k}}U^k - (\tilde{P}_{ijk}\tilde{U}^{\bar{k}} + \tilde{P}_{i\bar{j}\bar{k}}\tilde{U}^k) + (\tilde{R}_{i\bar{j}k\bar{l}} - R_{i\bar{j}k\bar{l}})v_{k,\bar{p}}v_{l,p}\right]w^iw^{\bar{j}} \\ &= -\frac{m}{2}(R_{i\bar{j}}v_i + R_{i\bar{i}}v_{\bar{j}}) + \frac{m}{2}R_{i\bar{j}p\bar{q}}v^pv^{\bar{q}} + \frac{m^2}{4}(nv_iv_{\bar{j}} - g_{i\bar{j}}|v|^2) + \frac{m}{2}(R_{i\bar{j},p}v^p + R_{i\bar{j},\bar{p}}v^{\bar{p}}) \\ &\quad + \frac{m_t}{2}G_{i\bar{j}k\bar{l}}v^kv^{\bar{l}} - \frac{m}{2}[R_{i\bar{j}}|v|^2 + R_{k\bar{j}}v_iv_{\bar{k}} + R_{i\bar{j}}v_kv_{\bar{j}}] + \frac{m_t}{2}(Rg_{i\bar{j}} + R_{i\bar{j}}) - \frac{m}{2}(RR_{i\bar{j}} + R_{i\bar{k}}R_{k\bar{j}}).\end{aligned}$$

我们有性质 2.1.

性质 2.1 设 $(g_{i\bar{j}}(x, t), X)$ 是方程 (1.1) 的解, 则对 $\forall (x, t) \in B_{r^2}(o, \frac{1}{2}e^{-\frac{n+1}{4}}r) \times (0, r^2]$, $\forall v \in T_x X$, 我们有

$$\text{CNS} \geq -CmM^2|w|^2.$$

证明 由

$$-\frac{m(t)}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}) \leq R_{i\bar{j}k\bar{l}} \leq \frac{M}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$$

可知,

$$-\frac{m(t)}{2}(n+1)g_{i\bar{j}} \leq R_{i\bar{j}} \leq \frac{M}{2}(n+1)g_{i\bar{j}}, \quad -\frac{m(t)}{2}(n+1)n \leq R \leq \frac{M}{2}(n+1)n.$$

由曲率假设条件和曲率的局部导数估计, 对 $\forall (x, t) \in B_{r^2}(o, \frac{1}{2}e^{-\frac{n+1}{4}}r) \times (0, r^2]$, 易见

$$\text{CNS} \geq (m'(t) - AmM)|v|^2|w|^2 - CmM^2|w|^2 \geq -CmM^2|w|^2.$$

3 局部 Harnack 估计

本节首先证明定理 1.1, 然后利用该定理得到两个推论.

我们将局部向量场 v 和 w 在 (x, t) 邻域内延拓, 满足

$$v_{l,k} = v_{\bar{k},\bar{l}} = 0, \quad w_{,j}^i = w_{,\bar{j}}^{\bar{i}} = 0, \\ \left(\frac{\partial}{\partial t} - \Delta\right)w^i = 0, \quad Y_{i\bar{j}} = 0, \quad U^k = 0.$$

记

$$\widetilde{M}_{k\bar{l}} = \widetilde{P}_{k\bar{l}i}w^i = (R_{i\bar{l},k} + \widetilde{R}_{i\bar{j}k\bar{l}}v^{\bar{j}})w^i, \\ \widetilde{M}_{kl} = \widetilde{P}_{k\bar{j}l}w^{\bar{j}} = (R_{k\bar{j},l} + \widetilde{R}_{i\bar{j}k\bar{l}}v^i)w^{\bar{j}}.$$

则

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)[(\widetilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}})w^iw^{\bar{j}}] &= \left(\frac{\partial}{\partial t} - \Delta\right)(\widetilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}})w^iw^{\bar{j}} \\ &= \widetilde{R}_{i\bar{j}k\bar{l}}(\widetilde{Q}_{l\bar{k}} + \varphi g_{l\bar{k}})w^iw^{\bar{j}} + |\widetilde{M}_{k\bar{l}}|^2 + |\widetilde{M}_{kl}|^2 \\ &\quad - \frac{1}{2}[R_{i\bar{k}}(\widetilde{Q}_{k\bar{j}} + \varphi g_{k\bar{j}}) + R_{k\bar{j}}(\widetilde{Q}_{i\bar{k}} + \varphi g_{i\bar{k}})]w^iw^{\bar{j}} + \widetilde{R}_{k\bar{l}}^{-1}\widetilde{H}_{i\bar{k}}\widetilde{H}_{l\bar{j}}w^iw^{\bar{j}} \\ &\quad + \left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi g_{i\bar{j}} - \varphi \widetilde{R}_{i\bar{j}}\right]w^iw^{\bar{j}} + \text{CNS}. \end{aligned}$$

这里 $\widetilde{H}_{i\bar{j}} = \widetilde{Q}_{i\bar{j}} - \frac{1}{2}(\widetilde{P}_{i\bar{j}k}v^k + \widetilde{P}_{i\bar{j}\bar{l}}v^{\bar{l}})$. 若 $-\frac{m(t)}{2}(n+1)g_{i\bar{j}} \leq R_{i\bar{j}} \leq \frac{M}{2}(n+1)g_{i\bar{j}}$, 求迹即得 $0 \leq \widetilde{R}_{i\bar{j}} \leq \frac{M+m}{2}(n+1)g_{i\bar{j}}$, 显然有 $\widetilde{R}_{i\bar{j}}^{-1} \geq \frac{2}{(M+m)(n+1)}g_{i\bar{j}}$, 易知

$$\begin{aligned} \widetilde{R}_{k\bar{l}}^{-1}\widetilde{H}_{i\bar{k}}\widetilde{H}_{l\bar{j}} &\geq \frac{2}{(M+m)(n+1)}g_{k\bar{l}}[-(\widetilde{H}_{i\bar{k}} + \varphi g_{i\bar{k}})(\widetilde{H}_{l\bar{j}} + \varphi g_{l\bar{j}}) \\ &\quad + \widetilde{H}_{i\bar{k}}(\widetilde{H}_{l\bar{j}} + \varphi g_{l\bar{j}}) + \widetilde{H}_{i\bar{k}}(\widetilde{H}_{i\bar{k}} + \varphi g_{i\bar{k}}) + \varphi^2 g_{i\bar{k}}g_{l\bar{j}}]. \end{aligned}$$

我们有

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)(\widetilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}})w^iw^{\bar{j}} &\geq \widetilde{R}_{i\bar{j}k\bar{l}}(\widetilde{Q}_{l\bar{k}} + \varphi g_{l\bar{k}})w^iw^{\bar{j}} + |\widetilde{M}_{k\bar{l}}|^2 + |\widetilde{M}_{kl}|^2 \\ &\quad - \frac{1}{2}[R_{i\bar{k}}(\widetilde{Q}_{k\bar{j}} + \varphi g_{k\bar{j}}) + R_{k\bar{j}}(\widetilde{Q}_{i\bar{k}} + \varphi g_{i\bar{k}})]w^iw^{\bar{j}} \\ &\quad + \frac{2}{(M+m)(n+1)}g_{k\bar{l}}[-(\widetilde{H}_{i\bar{k}} + \varphi g_{i\bar{k}})(\widetilde{H}_{l\bar{j}} + \varphi g_{l\bar{j}}) \\ &\quad + \widetilde{H}_{i\bar{k}}(\widetilde{H}_{l\bar{j}} + \varphi g_{l\bar{j}}) + \widetilde{H}_{i\bar{k}}(\widetilde{H}_{i\bar{k}} + \varphi g_{i\bar{k}}) + \varphi^2 g_{i\bar{k}}g_{l\bar{j}}] \\ &\quad + \left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi g_{i\bar{j}} - \varphi R_{i\bar{j}}\right]w^iw^{\bar{j}} + \text{CNS}. \end{aligned}$$

接下来我们将证明定理 1.1.

证明 设 $\varphi = (n+2)BM(\frac{1}{(r^2-d^2)^2} + \frac{1}{t})$, 若向量 v 首次使得 $\widetilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}}$ 为 0 时, 我们有

$$0 = \frac{\partial(\widetilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}})(V + s\widetilde{V})}{\partial s} \Big|_{s=0} = \widetilde{P}_{i\bar{j}k}v^k + \widetilde{P}_{i\bar{j}\bar{l}}v^{\bar{l}}, \quad \forall v \in T_x X.$$

经过直接的计算不难得到

$$\begin{aligned}\Delta\varphi &= (n+2)BM(24(r^2-d^2)^{-4}d^2 + 4(r^2-d^2)^{-3} + 4(r^2-d^2)^{-3}d\Delta d), \\ \frac{\partial}{\partial t}\varphi &= (n+2)BM\left(4(r^2-d^2)^{-3}d\frac{\partial d}{\partial t} - \frac{1}{t^2}\right),\end{aligned}$$

根据 [1] 中的引理 3.4.1, 设 $r_0 = \frac{d(x,0)}{2}$, 我们有

$$\begin{aligned}\left(\Delta - \frac{\partial}{\partial t}\right)\varphi &\leq (n+2)BM\left[4(r^2-d^2)^{-3} + 4(r^2-d^2)^{-3}d(2n-1)\left(\frac{2}{3}M\frac{d}{2} + \frac{2}{d}\right)\right. \\ &\quad \left.+ 24(r^2-d^2)^{-4}d^2 + t^{-2}\right] \\ &= (n+2)BM[C_0(r^2-d^2)^{-3} + 24(r^2-d^2)^{-4}d^2 + t^{-2}].\end{aligned}$$

由于 $Mr^2 = 1$, $M \geq 1$, $r^2 - d^2 \leq r^2 = \frac{1}{M}$, 容易知道, 当 B 足够大时,

$$(n+2)M\left(\Delta - \frac{\partial}{\partial t}\right)\varphi \leq (n+2)^2BM^2[C_1(r^2-d^2)^{-4} + t^{-2}] \leq \varphi^2.$$

另一方面,

$$\tilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}} - \frac{1}{2}(\tilde{P}_{i\bar{j}k}v^k + \tilde{P}_{i\bar{j}\bar{l}}v^{\bar{l}}) = \tilde{H}_{i\bar{j}} + \varphi g_{i\bar{j}} = 0,$$

由性质 2.1 可得

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)(\tilde{Q}_{i\bar{j}} + \varphi g_{i\bar{j}})w^i w^{\bar{j}} &\geq \frac{2}{(M+m)(n+1)} \varphi^2 g_{i\bar{j}} w^i w^{\bar{j}} \\ &\quad + \left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi g_{i\bar{j}} - \varphi R_{i\bar{j}}\right]w^i w^{\bar{j}} + \text{CNS} \\ &\geq \frac{2}{(M+m)(n+1)} \varphi^2 g_{i\bar{j}} w^i w^{\bar{j}} \\ &\quad + \left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi g_{i\bar{j}} - \varphi R_{i\bar{j}}\right]w^i w^{\bar{j}} - CmM^2 \\ &\geq \left[\frac{1}{M(n+1)}\varphi^2 - \frac{1}{M(n+2)}\varphi^2 - \frac{M(n+1)}{2}\varphi - CmM^2\right]|w|^2 \\ &\geq \left[\left(\frac{1}{M(n+1)(n+2)} - \epsilon\right)\varphi^2 - \frac{1}{\epsilon}\frac{M^2(n+1)^2}{4} - CmM^2\right]|w|^2 \\ &\geq [C_1B^2M^3 - C_2M^3]|w|^2 > 0\end{aligned}$$

这里 C_1, C_2 为仅依赖于 n 的常数, 且 B 足够大. 证毕.

我们接下来证明推论.

推论 1.1 的证明 对 $\tilde{Q}_{i\bar{j}}$ 求迹即得.

推论 1.2 的证明 在 (o, t) , $\forall v \in T_oX$, 由推论 1.1, 我们有

$$\begin{aligned}\frac{\partial R}{\partial t} + R_{,k}v^k + R_{,\bar{k}}v^{\bar{k}} + \left[R_{i\bar{j}} + \frac{m}{2}(n+1)g_{i\bar{j}}\right]v^i v^{\bar{j}} + \frac{m(n+1)}{2}R \\ + BM^2n(n+1)\left(\frac{1}{r^2} + \frac{1}{t}\right) \geq 0.\end{aligned}$$

则若 $v = 0$, 我们有

$$\frac{\partial R}{\partial t} + \frac{m(n+1)}{2}R + BM^2n(n+1)\left(\frac{1}{r^2} + \frac{1}{t}\right) \geq 0.$$

同时, 对 $\forall \lambda \in \mathbb{R}$, 我们有

$$\begin{aligned} & \lambda^2 \left[R_{i\bar{j}} + \frac{m}{2}(n+1)g_{i\bar{j}} \right] v^i v^{\bar{j}} + \lambda(R_{,k}v^k + R_{,\bar{k}}v^{\bar{k}}) + \frac{\partial R}{\partial t} + \frac{m(n+1)}{2}R \\ & + BM^2n(n+1)\left(\frac{1}{r^2} + \frac{1}{t}\right) \geq 0. \end{aligned}$$

显然有

$$\begin{aligned} |R_{,k}v^k + R_{,\bar{k}}v^{\bar{k}}|^2 & \leq 4 \left[R_{i\bar{j}} + \frac{m}{2}(n+1)g_{i\bar{j}} \right] v^i v^{\bar{j}} \left[\frac{\partial R}{\partial t} + \frac{m(n+1)}{2}R \right. \\ & \left. + BM^2n(n+1)\left(\frac{1}{r^2} + \frac{1}{t}\right) \right]. \end{aligned}$$

易知在 (o, r^2) , 我们有

$$\begin{aligned} & \frac{\partial R}{\partial t} + \frac{m(n+1)}{2}R + BM^2n(n+1)\left(\frac{1}{r^2} + \frac{1}{t}\right) \\ & = \Delta R + |\text{Ric}|^2 + \frac{m(n+1)}{2}R + BM^2n(n+1)\left(\frac{1}{r^2} + \frac{1}{t}\right) \\ & \leq CM^3. \end{aligned}$$

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Local Harnack estimate for Kähler-Ricci flow on complete Kähler manifold

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Abstract In this paper, we firstly give the evolving equation of the curvature under the unnormalized Kähler-Ricci flow. Then we get the extra term CNS which is determined by the Harnack quantity satisfying the local condition of the curvature. By using the estimate of CNS, we obtain the local Harnack estimate of Kähler-Ricci flow on the complete Kähler manifold. As a corollary, we get a sharp derivative estimate of scalar curvature.

Keywords: local Harnack estimate, Kähler-Ricci flow

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