含非完好界面的非线性弹性波超材料中SH波的非互易传输

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摘要 研究了含非完好界面的非线性弹性波超材料中的非互易传输现象. 通过结构的非对称性和材料的非线性打破了经典弹性波系统中的互易性原理. 入射的反平面剪切(shear horizontal, SH)波与材料的非线性效应相互作用产生二次谐波. 根据Bloch定理, 利用刚度矩阵法得到了基频波和二次谐波的带隙和透射系数. 通过与完好界面情况下的结果对比, 讨论了界面特性对层状周期结构中SH波的带隙和非互易传输的影响. 本研究有助于设计具有非互易传输性能的弹性波超材料器件.

关键词 非线性弹性波超材料, 非互易传输, 二次谐波, 非完好界面, 带隙

近些年来, 声子晶体和弹性波超材料作为一种人工周期结构受到了国内外研究者的广泛关注^[1-3]. 这些结构可以实现波的调控, 并且具有传统材料所不具备的特性, 如带隙^[4]、弹性波斗篷^[5]和负折射^[6]等. 在实际应用中, 上述波动现象有助于设计先进器件. 其中一个众所周知的特性为弹性波带隙, 即在特定的频率范围内弹性波无法进行传播. 基于这一特性, 能够实现许多工程应用, 例如滤波器以及声振隔离等.

材料非线性的存在, 使得非线性波将带来新的现象和有趣的传播行为, 因此受到了越来越多研究者的广泛关注^[7-9]. 我们知道, 非线性现象一个重要的特性是高阶谐波的产生. Deng和Xiang^[10]研究了层状结构中基于SH波与材料非线性相互作用产生的非线性Lamb波的传输特性. Liang等人^[11]结合声子晶体的带隙特性和声学材料非线性实现了声波二极管. 弹性波二极管

或非互易传输是一种新的现象, 即弹性波在一个方向可以传播, 但是在另一个方向则被截止^[12-14].

周期结构的研究通常是建立在完好界面的基础上, 即位移和应力分量在界面处是连续的. 以往的研究工作中, 讨论了入射SH波在含完好界面的非线性弹性波超材料中的非互易传输^[15]. 然而在实际应用中, 层状结构中相邻层间的界面可呈现非完好特征. 理论研究表明, 非完好界面对结构的带隙和频散关系的影响显著^[16,17]. 由于弹性波的非互易传输现象依赖于结构的带隙, 因此研究非完好界面对非互易传输的影响将显得十分必要. 本文结合声子晶体的带隙特性与材料的非线性实现了含非完好界面结构中SH波的非互易传输, 讨论了非完好界面对弹性波传输特性的影响, 实现了其对非线性声子晶体和弹性波超材料中带隙和非互易传输的调控.

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1 模型和波动方程

图1(a)给出了由两种不同的非线性材料A和B周期排列而组成的非线性声子晶体. 该结构由 m 个单胞构成, 每个单胞的厚度为 $d=d_1+d_2$, 其中 d_1 和 d_2 表示子胞的厚度. 局部坐标系如图1(a)所示, 其中 $\hat{x}$, $\hat{y}$ 和 $\hat{z}$ 为坐标系的单位矢量. 结构中两个相邻固体层黏结而形成界面, 本文假设界面为非完好界面, 即材料黏结面两侧物理量不连续^[17]. 将非完好界面等效为力学弹簧模型(图1(b)), 即在界面处应力连续而位移不连续. 界面性质由法向和切向界面刚度系数 K_N 和 K_T 表示.

非线性弹性波超材料的理论模型如图1(c)所示, 该结构由两种线性材料周期交替而成的声子晶体和非线性介质A组成. 通过引入非线性机理打破了整体结构的对称性. 利用声子晶体的带隙特性和材料的非线性实现了弹性波的非互易传输, 即弹性波沿正向可以传播, 但是反向将被截止, 如图1(d)所示.

考虑以 x 轴为极化方向的入射SH波沿 yoz 平面内传播. 由于存在材料的非线性, 根据二阶微扰近似, 可得基频波和二次谐波的控制方程^[10]为

$$\begin{aligned} \rho_j \frac{d\mathbf{u}_j^{(1)}}{dt^2} - (\lambda_j + 2\mu_j) \nabla (\nabla \cdot \mathbf{u}_j^{(1)}) \\ + \mu_j \nabla \times (\nabla \times \mathbf{u}_j^{(1)}) = 0, \end{aligned} \quad (1)$$

$$\begin{aligned} \rho_j \frac{d\mathbf{u}_j^{(2)}}{dt^2} - (\lambda_j + 2\mu_j) \nabla (\nabla \cdot \mathbf{u}_j^{(2)}) \\ + \mu_j \nabla \times (\nabla \times \mathbf{u}_j^{(2)}) = \mathbf{F}(\mathbf{u}_j^{(1)}), \end{aligned} \quad (2)$$

其中, $\mathbf{u}_j^{(1)}$ 和 $\mathbf{u}_j^{(2)}$ 为基频波和二次谐波的矢量位移; 下标 j 表示周期层状结构的子胞; ρ_j 为材料的密度; λ_j 和 μ_j 为二阶弹性常数; $\nabla = (\partial/\partial x_j, \partial/\partial y_j, \partial/\partial z_j)$ 为Nabla算子; t 为时间; $\mathbf{F}(\mathbf{u}_j^{(1)})$ 为二次谐波彻体力, 由基频波和材料的非线性相互作用而产生^[10,16].

通过求解方程(1)得到基频波的位移解为

$$\begin{aligned} \mathbf{u}_j^{(1)} = \hat{x} \left\{ u_{jT_1}^{(1)} \exp[ik^{(1)}(\alpha_{jT}y + z) - i\omega t] \right. \\ \left. + u_{jT_2}^{(1)} \exp[ik^{(1)}(-\alpha_{jT}y + z) - i\omega t] \right\}, \end{aligned} \quad (3)$$

其中, $u_{jT_1}^{(1)}$ 和 $u_{jT_2}^{(1)}$ 为基频波的振幅; $i = \sqrt{-1}$; $k^{(1)} = \omega/c^{(1)}$; ω 和 $c^{(1)}$ 分别为频率和入射波的相速度; $\alpha_{jT} = \sqrt{c^{(1)}/c_{jT} - 1}$; c_{jT} 为横波的波速. 后续计算中, 为简化推导将省略公共项 $\exp(-i\omega t)$.

初始SH波在结构中传播, 与材料非线性相互作用可产生二次谐波彻体力. 将基频位移表达式(3)代入方程式(2)中, 可得二次谐波彻体力^[10]为

图1 (网络版彩色)SH波入射时, 含非完好界面的非线性声子晶体和弹性波超材料的力学模型. (a) 非线性声子晶体及局部坐标系; (b) 非完好界面模型; (c) 非线性弹性波超材料; (d) 非互易传输特性

Figure 1 (Color online) The nonlinear phononic crystal and elastic wave metamaterial with imperfect interfaces for an incident SH wave. (a) The nonlinear phononic crystal and the local coordinates; (b) the imperfect interfacial model; (c) the nonlinear elastic wave metamaterial; (d) its nonreciprocal transmission

$$\begin{aligned} F(u_j^{(1)}) = & -i \left(\lambda_j + 2\mu_j + \frac{A_j}{2} + B_j \right) K_{jT}^3 (u_{jT_1}^{(1)})^2 \bar{K}_{jT_1}^{(1)} \exp[2ik^{(1)}(\alpha_{jT}y + z)] \\ & -i \left(\lambda_j + 2\mu_j + \frac{A_j}{2} + B_j \right) K_{jT}^3 (u_{jT_2}^{(1)})^2 \bar{K}_{jT_2}^{(1)} \exp[2ik^{(1)}(-\alpha_{jT}y + z)] \\ & -2i\bar{z} \left\{ \mu_j + \frac{A_j}{4} - \left[1 - 2 \left(\frac{c_{jT}}{c^{(1)}} \right)^2 \right] \left(\lambda_j + \mu_j + \frac{A_j}{4} + B_j \right) \right\} u_{jT_1}^{(1)} u_{jT_2}^{(1)} k^{(1)} K_{jT}^2 \exp(2ik^{(1)}z), \end{aligned} \quad (4)$$

其中, $\bar{K}_{jT_1}^{(1)} = k^{(1)}(\alpha_{jT}\hat{y} + \hat{z})$ 和 $\bar{K}_{jT_2}^{(1)} = k^{(1)}(-\alpha_{jT}\hat{y} + \hat{z})$ 为基频位移分量的波矢量; $K_{jT} = |\bar{K}_{jT_1}^{(1)}| = |\bar{K}_{jT_2}^{(1)}|$; A_j 和 B_j 为三阶弹性常数.

结合式(4)可以得到:

$$\bar{x} \cdot F(u_j^{(1)}) = 0. \quad (5)$$

由此可以推断出, 作为二次谐波的驱动源, 将会产生一系列平面内二次谐波沿 yoz 平面进行传播. 结合控制方程(2)可得二次谐波位移解为

$$\begin{aligned} u_{jT}^{(2)} = & (\bar{K}_{jT_1}^{(2)} \times \bar{x}) u_{jT_1}^{(2)} \exp[ik^{(2)}(\beta_{jT}y + z)] \\ & + (\bar{K}_{jT_2}^{(2)} \times \bar{x}) u_{jT_2}^{(2)} \exp[ik^{(2)}(-\beta_{jT}y + z)] \\ & + \bar{K}_{jL_1}^{(2)} u_{jL_1}^{(2)} \exp[ik^{(2)}(\beta_{jL}y + z)] \\ & + \bar{K}_{jL_2}^{(2)} u_{jL_2}^{(2)} \exp[ik^{(2)}(-\beta_{jL}y + z)], \end{aligned} \quad (6)$$

其中, $\bar{K}_{jQp}^{(2)} (Q=T, L; p=1, 2)$ 为二次谐波位移分量的波矢量; $u_{jQp}^{(2)}$ 为待解的振幅; $\beta_{jQ} = \sqrt{c^{(2)}/c_{jQ} - 1}$; $c^{(2)}$ 为二次谐波入射波的相速度; c_{jL} 为纵波的波速; $k^{(2)}$ 为二次谐波的波数.

2 含非完好界面的刚度矩阵

根据本构关系, 可以写出剪切应力表达式为

$$\tau_{jxy}^{(1)} = \mu_j \frac{\partial u_j^{(1)}}{\partial y}. \quad (7)$$

将基频波位移解(式(3))代入上式得

$$\begin{aligned} \tau_{jxy}^{(1)} = & i\mu_j k^{(1)} \alpha_{jT} u_{jT_1}^{(1)} \exp[ik^{(1)}(\alpha_{jT}y + z)] \\ & - i\mu_j k^{(1)} \alpha_{jT} u_{jT_2}^{(1)} \exp[ik^{(1)}(-\alpha_{jT}y + z)]. \end{aligned} \quad (8)$$

完好界面是指力学参数在界面处均连续的界面. 如果部分或全部力学参数在界面处不连续则称之为非完好界面. 在具体计算中, 可将非完好界面假设为一种无厚度的薄层^[17]. 本文将非完好界面等效为力学弹簧

模型, 即应力分量在边界处连续而位移分量不连续. 因此, 基频波在界面 $2n$ 和 $2n+1$ 处所满足的边界条件为

$$\begin{aligned} \tau_{jxy}^{(1)} \Big|_{y=2n^+/2n+1^+} \\ = \tau_{jxy}^{(1)} \Big|_{y=2n^-/2n+1^-} \\ = \tau_{jxy}^{(1)} \Big|_{y=2n/2n+1}, \\ \tau_{jxy}^{(1)} \Big|_{y=2n/2n+1} \\ = K_T \left(u_j^{(1)} \Big|_{y=2n^+/2n+1^+} - u_j^{(1)} \Big|_{y=2n^-/2n+1^-} \right). \end{aligned} \quad (9)$$

根据本构关系, 子胞左右边界的状态向量可由刚度矩阵表示为如下形式:

$$\begin{aligned} \begin{Bmatrix} \tau_{jxy}^{(1)} \Big|_{y=2n-1} \\ \tau_{jxy}^{(1)} \Big|_{y=2n^-} \end{Bmatrix} &= \mathbf{K}^A \begin{Bmatrix} u_j^{(1)} \Big|_{y=2n-1} \\ u_j^{(1)} \Big|_{y=2n^-} \end{Bmatrix}, \\ \begin{Bmatrix} \tau_{jxy}^{(1)} \Big|_{y=2n^+} \\ \tau_{jxy}^{(1)} \Big|_{y=2n+1} \end{Bmatrix} &= \mathbf{K}^B \begin{Bmatrix} u_j^{(1)} \Big|_{y=2n^+} \\ u_j^{(1)} \Big|_{y=2n+1} \end{Bmatrix}, \end{aligned} \quad (10)$$

其中, $\mathbf{K}^W (W=A, B)$ 为子胞的刚度矩阵, 其分量为

$$\begin{aligned} K_{11}^W &= \frac{-i\mu_j \omega \alpha_{jT} (rd_{j1} + rd_{j2})}{c^{(1)}(rd_{j1} - rd_{j2})}, \\ K_{12}^W &= \frac{2i\mu_j \omega \alpha_{jT}}{c^{(1)}(rd_{j1} - rd_{j2})}, \\ K_{21}^W &= \frac{-2i\mu_j \omega \alpha_{jT} rd_{j1} rd_{j2}}{c^{(1)}(rd_{j1} - rd_{j2})}, \\ K_{22}^W &= \frac{i\mu_j \omega \alpha_{jT} (rd_{j1} + rd_{j2})}{c^{(1)}(rd_{j1} - rd_{j2})}, \\ rd_{j1} &= \exp(i\omega \alpha_{jT} d_j / c^{(1)}), \\ rd_{j2} &= \exp(-i\omega \alpha_{jT} d_j / c^{(1)}). \end{aligned} \quad (11)$$

将式(3)和(8)代入边界条件式(9), 可得刚度矩阵 $\mathbf{K}^I$ 为

$$\mathbf{K}^I = \begin{bmatrix} -K_T & K_T \\ -K_T & K_T \end{bmatrix}. \quad (12)$$

联立式(10), (11)和(12), 可得一个单胞的局部刚度矩阵为

$$\mathbf{K} = \begin{bmatrix} K_{11}^{AIB} + K_{12}^{AIB}(K_{11}^I - K_{22}^{AIB})^{-1}K_{21}^{AIB} & -K_{12}^{AIB}(K_{11}^I - K_{22}^{AIB})^{-1}K_{12}^I \\ K_{21}^I(K_{11}^I - K_{22}^{AIB})^{-1}K_{21}^{AIB} & K_{22}^I - K_{21}^I(K_{11}^I - K_{22}^{AIB})^{-1}K_{12}^I \end{bmatrix}, \quad (13)$$

其中,

$$\begin{aligned} K_{11}^{AIB} &= K_{11}^{AI} + K_{12}^{AI}(K_{11}^B - K_{22}^{AI})^{-1}K_{21}^{AI}, & K_{12}^{AIB} &= -K_{12}^{AI}(K_{11}^B - K_{22}^{AI})^{-1}K_{12}^B, \\ K_{21}^{AIB} &= K_{21}^B(K_{11}^B - K_{22}^{AI})^{-1}K_{21}^{AI}, & K_{22}^{AIB} &= K_{22}^B - K_{21}^B(K_{11}^B - K_{22}^{AI})^{-1}K_{12}^{AI}, \\ K_{11}^{AI} &= K_{11}^A + K_{12}^A(K_{11}^I - K_{22}^A)^{-1}K_{21}^A, & K_{12}^{AI} &= -K_{12}^A(K_{11}^I - K_{22}^A)^{-1}K_{12}^I, \\ K_{21}^{AI} &= K_{21}^I(K_{11}^I - K_{22}^A)^{-1}K_{21}^A, & K_{22}^{AI} &= K_{22}^I - K_{21}^I(K_{11}^I - K_{22}^A)^{-1}K_{12}^A. \end{aligned} \quad (14)$$

由于结构在y轴方向上具有周期性, 根据Bloch定理, 可得应力和位移分量的关系如下:

$$\begin{Bmatrix} u_j^{(1)} \\ \tau_{jxy}^{(1)} \end{Bmatrix}_{2n+1} = e^{ikd} \begin{Bmatrix} u_j^{(1)} \\ \tau_{jxy}^{(1)} \end{Bmatrix}_{2n-1}. \quad (15)$$

通过矩阵运算可得传递矩阵 $\mathbf{T}$ 的表达式为

$$\mathbf{T} = \begin{bmatrix} -K_{12}^{-1}K_{11} & K_{12}^{-1} \\ K_{21} - K_{22}K_{12}^{-1}K_{11} & K_{22}K_{12}^{-1} \end{bmatrix}. \quad (16)$$

$$\mathbf{K}^m = \begin{bmatrix} K_{11}^{m-1} + K_{12}^{m-1}(K_{11}^M - K_{22}^{m-1})^{-1}K_{21}^M & -K_{12}^{m-1}(K_{11}^M - K_{22}^{m-1})^{-1}K_{12}^M \\ K_{21}^M(K_{11}^M - K_{22}^{m-1})^{-1}K_{21}^{m-1} & K_{22}^M + K_{12}^M(K_{11}^M - K_{22}^{m-1})^{-1}K_{12}^{m-1} \end{bmatrix}. \quad (18)$$

对于入射SH波, 在层状周期结构的入射边界处的位移场由一个入射波和一个反射波构成, 位移解可以表示为

$$\begin{aligned} u_{(1)} &= A_I \exp[ik^{(1)}(\alpha_{(1)T}y + z)] \\ &+ A_R \exp[ik^{(1)}(-\alpha_{(1)T}y + z)], \end{aligned} \quad (19)$$

其中, 下标(1)表示入射边界, A_I 和 A_R 分别为入射波和反射波的振幅.

相应的透射边界处位移为

$$u_{(2)} = A_T \exp[ik^{(1)}(\alpha_{(2)T}y + z)], \quad (20)$$

其中, 下标(2)表示透射边界, A_T 为透射波的振幅.

可得含有应力和位移的状态向量的整体刚度矩阵为

$$\begin{Bmatrix} \tau_{(1)xy} \\ \tau_{(2)xy} \end{Bmatrix} = \mathbf{K}^m \begin{Bmatrix} u_{(1)} \\ u_{(2)} \end{Bmatrix}. \quad (21)$$

结合式(15)和(16)可得系统的特征方程:

$$|\mathbf{T} - e^{ikd}\mathbf{I}| = 0. \quad (17)$$

通过求解特征值方程式(17), 可以得到系统的带隙特征. 当存在实数解时, 则表示在该频率范围内有弹性波能够进行传播, 即为通带, 反之则为禁带.

考虑具有 m 个单胞的层状周期结构, 假设 $\mathbf{K}^{m-1}$ 为前 $m-1$ 个单胞的刚度矩阵, $\mathbf{K}^M$ 为第 m 个单胞的刚度矩阵, 根据迭代法可以得到周期结构的整体刚度矩阵 $\mathbf{K}^m$ 为^[18]

因此, 可以得到透射系数和反射系数:

$$\begin{Bmatrix} F_{DTT} \\ F_{RTT} \end{Bmatrix} = \mathbf{M}^{-1}\mathbf{N}, \quad (22)$$

其中, $F_{DTT}=A_T/A_I$ 和 $F_{RTT}=A_R/A_I$ 分别为透射系数和反射系数, 系数矩阵 $\mathbf{M}$ 和 $\mathbf{N}$ 中的元素为

$$\begin{aligned} M_{11} &= K_{12}^m, \\ M_{12} &= K_{11}^m + ik^{(1)}\alpha_{(1)T}\mu_{(1)}, \\ M_{21} &= K_{22}^m - ik^{(1)}\alpha_{(2)T}\mu_{(2)}, \\ M_{22} &= K_{21}^m, \\ N_{11} &= ik^{(1)}\alpha_{(1)T}\mu_{(1)} - K_{11}^m, \\ N_{21} &= -K_{21}^m. \end{aligned} \quad (23)$$

基频波与材料非线性相互作用将产生二次谐波, 其位移表达式(6)可分解为y轴和z轴两个方向的分量:

$$\begin{aligned} u_{jy}^{(2)} &= -\sin\theta_{jT} u_{jT_1}^{(2)} sd_{j1} + \sin\theta_{jT} u_{jT_2}^{(2)} sd_{j2} \\ &\quad + \cos\theta_{jL} u_{jL_1}^{(2)} sd_{j3} - \cos\theta_{jL} u_{jL_2}^{(2)} sd_{j4}, \\ u_{jz}^{(2)} &= \cos\theta_{jT} u_{jT_1}^{(2)} sd_{j1} + \cos\theta_{jT} u_{jT_2}^{(2)} sd_{j2} \\ &\quad + \sin\theta_{jL} u_{jL_1}^{(2)} sd_{j3} + \sin\theta_{jL} u_{jL_2}^{(2)} sd_{j4}, \end{aligned} \quad (24)$$

其中,

$$\begin{aligned} sd_{j1} &= \exp[ik^{(2)}(\beta_{jT}y + z)], \\ sd_{j2} &= \exp[ik^{(2)}(-\beta_{jT}y + z)], \\ sd_{j3} &= \exp[ik^{(2)}(\beta_{jL}y + z)], \\ sd_{j4} &= \exp[ik^{(2)}(-\beta_{jL}y + z)], \\ \sin\theta_{jQ} &= c_{jQ}/c^{(2)}, \\ \cos\theta_{jQ} &= \beta_{jQ}\sin\theta_{jQ}. \end{aligned} \quad (25)$$

应力分量可以表示为

$$\begin{aligned} \tau_{jzy}^{(2)} &= \mu_j \left(\frac{\partial u_{jy}^{(2)}}{\partial z} + \frac{\partial u_{jz}^{(2)}}{\partial y} \right), \\ \sigma_{jyy}^{(2)} &= (\lambda_j + 2\mu_j) \frac{\partial u_{jy}^{(2)}}{\partial y} + \lambda_j \frac{\partial u_{jz}^{(2)}}{\partial z}. \end{aligned} \quad (26)$$

上述分量的详细表达式可参见文献[15]. 根据非完好界面的力学性质, 二次谐波在边界处所满足的边界条件为

$$\begin{aligned} \tau_{jzy}^{(2)} \Big|_{y=2n^+/2n+1^+} &= \tau_{jzy}^{(2)} \Big|_{y=2n^-/2n+1^-} = \tau_{jzy}^{(2)} \Big|_{y=2n/2n+1}, \\ \sigma_{jyy}^{(2)} \Big|_{y=2n^+/2n+1^+} &= \sigma_{jyy}^{(2)} \Big|_{y=2n^-/2n+1^-} = \sigma_{jyy}^{(2)} \Big|_{y=2n/2n+1}, \\ \tau_{jzy}^{(2)} \Big|_{y=2n/2n+1^-} &= K_T \left[u_{jz}^{(2)} \Big|_{y=2n^+/2n+1^+} - u_{jz}^{(2)} \Big|_{y=2n^-/2n+1^-} \right], \\ \sigma_{jyy}^{(2)} \Big|_{y=2n/2n+1^-} &= K_N \left[u_{jy}^{(2)} \Big|_{y=2n^+/2n+1^+} - u_{jy}^{(2)} \Big|_{y=2n^-/2n+1^-} \right]. \end{aligned} \quad (27)$$

由式(24)~(27), 可得二次谐波子胞和界面的刚度矩阵为

$$\begin{aligned} \begin{Bmatrix} \sigma^{(2)} \Big|_{y=2n-1} \\ \sigma^{(2)} \Big|_{y=2n^-} \end{Bmatrix} &= \mathbf{K}^{(2)A} \begin{Bmatrix} u^{(2)} \Big|_{y=2n-1} \\ u^{(2)} \Big|_{y=2n^-} \end{Bmatrix}, \\ \begin{Bmatrix} \sigma^{(2)} \Big|_{y=2n^-} \\ \sigma^{(2)} \Big|_{y=2n^+} \end{Bmatrix} &= \mathbf{K}^{(2)I} \begin{Bmatrix} u^{(2)} \Big|_{y=2n^-} \\ u^{(2)} \Big|_{y=2n^+} \end{Bmatrix}, \\ \begin{Bmatrix} \sigma^{(2)} \Big|_{y=2n^+} \\ \sigma^{(2)} \Big|_{y=2n+1^-} \end{Bmatrix} &= \mathbf{K}^{(2)B} \begin{Bmatrix} u^{(2)} \Big|_{y=2n^+} \\ u^{(2)} \Big|_{y=2n+1^-} \end{Bmatrix}, \\ \begin{Bmatrix} \sigma^{(2)} \Big|_{y=2n+1^-} \\ \sigma^{(2)} \Big|_{y=2n+1^+} \end{Bmatrix} &= \mathbf{K}^{(2)l} \begin{Bmatrix} u^{(2)} \Big|_{y=2n+1^-} \\ u^{(2)} \Big|_{y=2n+1^+} \end{Bmatrix}, \end{aligned} \quad (28)$$

其中, $\sigma^{(2)} = \{\tau_{jzy}^{(2)} \quad \sigma_{jyy}^{(2)}\}^T$ 和 $u^{(2)} = \{u_{jy}^{(2)} \quad u_{jz}^{(2)}\}^T$ 分别为应力和位移的状态向量; $\mathbf{K}^{(2)A}$ 和 $\mathbf{K}^{(2)B}$ 为子胞的刚度矩阵; $\mathbf{K}^{(2)l}$ 为界面的刚度矩阵, 可表示为

$$\mathbf{K}^{(2)l} = \begin{bmatrix} -K_N & 0 & K_N & 0 \\ 0 & -K_T & 0 & K_T \\ -K_N & 0 & K_N & 0 \\ 0 & -K_T & 0 & K_T \end{bmatrix}. \quad (29)$$

采用相似的推导过程, 可得二次谐波作用时一个单胞的局部刚度矩阵 $\mathbf{K}^{(2)}$ 以及 m 个单胞的整体刚度矩阵 $\mathbf{K}^{(2)m}$. 根据Bloch定理, 可得二次谐波情况的特征方程, 并可求解含非完好界面结构的非线性带隙特性.

二次谐波横波模态在入射边界处由一个入射波和两个反射波组成, 并可表示为

$$\begin{aligned} u_{(1)y} &= -\sin\theta_{(1)T} sd_{(1)1} + \sin\theta_{(1)T} sd_{(1)2} H_{RTT} \\ &\quad - \cos\theta_{(1)L} sd_{(1)4} H_{RTL}, \\ u_{(1)z} &= \cos\theta_{(1)T} sd_{(1)1} + \cos\theta_{(1)T} sd_{(1)2} H_{RTT} \\ &\quad + \sin\theta_{(1)L} sd_{(1)4} H_{RTL}, \end{aligned} \quad (30)$$

其中, H_{RTT} 和 H_{RTL} 分别为横波和纵波模态的反射系数.

透射边界处的位移为

$$\begin{aligned} u_{(2)y} &= -\sin\theta_{(2)T} sd_{(2)1} H_{DTT} \\ &\quad + \cos\theta_{(2)L} sd_{(2)3} H_{DTL}, \\ u_{(2)z} &= \cos\theta_{(2)T} sd_{(2)1} H_{DTT} \\ &\quad + \sin\theta_{(2)L} sd_{(2)3} H_{DTL}, \end{aligned} \quad (31)$$

其中, H_{DTT} 和 H_{DTL} 分别为横波和纵波模态的透射系数.

由此可求得二次谐波情况时的反射和透射系数, 并计算出二次谐波的传输特性. 本文将入射波为二次谐波纵波模态得到的透射系数定义为 H_{DLT} 和 H_{DLL} .

3 数值模拟和讨论

图1(a)中的非线性声子晶体由两种非线性材料组成, 分别为5052铝合金(A)和304不锈钢(B)^[10,19]. 如图1(c)所示, 非线性弹性波超材料由声子晶体和非线性介质(A)组合而成, 其中声子晶体由线性材料AZ31B镁合金(C)和TC4钛合金(D)组成^[20]. 本文采用切向和法向的界面刚度系数分别为 $K_T = 0.638 \times 10^{14} \text{ N/m}^3$ 和 $K_N = 3.175 \times 10^{14} \text{ N/m}^3$ ^[16,17], 入射波的相速度 $c^{(1)} = c^{(2)} =$

图2 (网络版彩色)含非完好界面的非线性声子晶体中基频波的带隙(a)和相应的透射系数(b)

Figure 2 (Color online) Band gap (a) and the transmission coefficient (b) of the fundamental wave in the nonlinear phononic crystal with imperfect interfaces

5000 m/s, 厚度比 $d_1/d_2=1/1$. 对于周期层状结构, 假设其包含 $m=20$ 个单胞.

图2给出了含非完好界面的非线性声子晶体中基频波的带隙和透射系数, 图中深色阴影区域表示其与完好界面结构带隙的对比结果. 从图2(a)可以看出, 基频波的带隙出现在频率区间 $0.22\sim 0.40$ 和 $0.55\sim 0.70$ MHz内. 与完好界面情况相比较, 在非完好界面条件下带隙的中心频率明显向低频区域移动, 并且第二阶带隙的宽度减小. 图2(b)表示相应的透射系数, 当实数解为零时对应于禁带, 反之则为通带. 结果表明, 带隙的位置和宽度与图2(a)中的结果完全吻合.

图3给出了含非完好界面的非线性声子晶体中二次谐波的带隙和透射系数. 可以看出, 图3(b)和(c)中透射系数为零的频率区间与图3(a)中带隙的位置吻合较好. 此外, 二次谐波带隙的变化与基频波的情况一致, 并且第二阶带隙宽度的变化更加明显. 完好界面时的禁带由于界面特性的影响转变为通带. 因此, 非完好界面可以作为一种调节周期层状结构中带隙位置和宽度的有效方法.

图4给出了含非完好界面的非线性弹性波超材料中基频波和二次谐波的带隙. 如图所示, 频率区间

图3 (网络版彩色)含非完好界面的非线性声子晶体中二次谐波的带隙和相应的透射系数. (a) 带隙; (b) 二次横波所产生的横波和纵波; (c) 二次纵波所产生的横波和纵波的透射系数

Figure 3 (Color online) Band gap and transmission coefficients of the second harmonic in the nonlinear phononic crystal with imperfect interfaces. (a) The band gap; the transmission coefficients of the generated transverse and longitudinal waves for the incident transverse wave (b) and the incident longitudinal wave (c)

图4 (网络版彩色)含非完好界面的非线性弹性波超材料中基频波(a)和二次谐波(b)的带隙

Figure 4 (Color online) Band gaps of the fundamental wave (a) and the second harmonic (b) in the nonlinear elastic wave metamaterial with imperfect interfaces

0.21~0.38, 0.56~0.63和0.81~0.96 MHz为基频波的禁带; 而0.13~0.18, 0.31~0.32和0.44~0.49 MHz则属于二次谐波的禁带。当基频波处于禁带而二次谐波处于通带时, 弹性波的非互易传输现象可以实现。因此, 频率区间0.21~0.31和0.32~0.38 MHz可视为非完好界面条件下的非互易传输区间。作为对比算例, 计算出含完好界面周期结构的非互易区间为0.23~0.31和0.34~0.40 MHz。可以明显看出, 界面特性使得非互易传输区间的中心频率向低频区域移动。

当入射SH波从结构右边界入射时, 与材料的非线性相互作用将产生二次谐波。此时系统中存在两种类型的弹性波从右向左传播。如图5所示, 在非互易区间0.21~0.31和0.32~0.38 MHz内, 基频波的透射系数为零而二次谐波的透射系数不为零。因此, 二次谐波正向可以进行传播。当入射SH波从结构左边界入射时, 由于线性材料不会产生非线性弹性波, 二次谐波的透射系数在整个频率区间内为零, 使得反向没有弹性波可以通过, 实现了非互易传输现象, 即弹性波在一个方向可以传播而另一个方向则被截止。

4 总结

本文研究了含非完好界面的非线性声子晶体和弹性波超材料的传输行为。考虑SH波入射情况且在结构中传播时与材料非线性相互作用产生二次谐波, 同时结合带隙特性实现了弹性波的非互易传输。进一步讨论了非完好界面对带隙和非互易行为的影响。研究结果表明, 与完好界面情况相比, 含有非完好界面的结构中带隙和非互易传输的中心频率明显向低频区域移动。本文研究有助于设计具有可调节功能的弹性波非互易传输器件。

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图5 (网络版彩色)入射波从右边界入射时, 含非完好界面的非线性弹性波超材料的非互易传输特性。(a) 基频波; 二次谐波(b)和二次纵波(c)所产生的横波和纵波的透射系数
Figure 5 (Color online) Nonreciprocal transmission in the nonlinear elastic wave metamaterial with imperfect interfaces for the incident wave from the right boundary. (a) The fundamental wave; the transmission coefficients of the generated transverse and longitudinal waves for the incident transverse wave (b) and the incident longitudinal wave (c) with double frequency

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Summary for “含非完好界面的非线性弹性波超材料中SH波的非互易传输”

Nonreciprocal transmission for the SH wave in a nonlinear elastic wave metamaterial with imperfect interfaces

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During the past decades, phononic crystals and elastic wave metamaterials as the artificial periodic structures have received considerable attention. Due to their distinguishing properties, these structures can be used for the wave energy manipulation, e.g., band gaps, elastic wave cloaking and topological state, etc. A well-known characteristic is the band gap which means that the elastic wave cannot propagate in some certain frequency regions. Based on this property, many engineering applications can be achieved, e.g., filtering and wave isolation, etc. The investigations mentioned above are mainly focused on the linear elastic problems which cannot present the nonlinear wave properties. However, the nonlinearity appearing in the materials and structures can bring new and interesting wave phenomena. One of the nonlinear characteristics is the generation of the higher order harmonics. It can be applied to the nonlinear nondestructive testing. Recently, increasing attention has been focused on the nonlinear wave motion, which can result in the nonlinear band gap and acoustic diode.

The new concept named as the acoustic diode has been presented by the combination of the band gap and material nonlinearity, which permits the wave propagation in only one direction. It can be achieved when the fundamental wave sits in the stop band but the second harmonic belongs to the pass band. However, the acoustic wave has one direction displacement component and usually propagates in the air or liquid. Thus we wonder whether the nonreciprocal phenomenon of the elastic wave can be achieved in the solid materials. The problem will become more interesting and necessary because of the coupling displacements with the vector characteristic in solids. In our previous work, we have discussed the nonreciprocal transmission for the incident SH and in-plane elastic waves in a nonlinear elastic wave metamaterial. The interface in the consecutive layers is always assumed as perfect, which means that the displacement and stress components are continuous across the interface. But perfect interfaces may become imperfect in practice during the manufacture, which indicates that the interfacial property on the band gaps is obvious. Then the imperfect interface can offer a new opportunity to tune the nonreciprocal transmission of the elastic wave which depends on the band gap property.

In this work, the nonreciprocal transmission for the SH wave in a layered nonlinear elastic wave metamaterial with the imperfect interfaces is investigated. The combination of the structural asymmetry and material nonlinearity breaks the inherent reciprocity of the classic wave system. The second harmonic can be generated by the interaction between the incident SH wave and material nonlinearity. Based on the Bloch's law and stiffness matrix method, the band gaps and transmission coefficients of both the fundamental and second harmonic waves are obtained. The effects of the interfacial properties on the nonlinear phononic crystal and elastic wave metamaterial are discussed. We find that comparing with the perfect interface system, the central frequency of the nonreciprocal regions shifts towards the low frequency region by the effects of imperfect interfaces. This present work is expected to be helpful to design the practical devices with the tuning nonreciprocal transmission of the elastic wave.

nonlinear elastic wave metamaterial, nonreciprocal transmission, second harmonic, imperfect interface, band gap

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